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SS2010

ÜBUNGEN ZUR VORLESUNG POSITIVE POLYNOME

BLATT 05

These exercises will be collected Tuesday 1st June in the mailbox n.14 of the Mathematics department.

In the following exercise we complete the proof of Hilbert's Theorem for any real closed field R.

1. Let R be a real closed field.

(i) (*Spectral theorem.*) Let $n \in \mathbb{N}$ and $M_n(R)$ the set of $n \times n$ matrices with coefficients in R. Show that for every symmetric matrix $A \in M_n(R)$ there is a diagonal matrix $D \in M_n(R)$ and $S \in M_n(R)$ such that

$$S^T S = I \quad \text{and} \quad A = SDS^T.$$

(Write details for $n = 2$ and explain in words why it is true $\forall\, n \in \mathbb{N}$).

(ii) Show that

$$f \in \mathcal{P}_{n,2d} \quad f = f_1^2 + \cdots + f_s^2 \;\Rightarrow\; \exists \text{ such an } s \text{ with } s \leqslant \binom{n+d}{d}.$$

(iii) Show that $\forall\, n \in \mathbb{N}$ and $\forall\, m \in \mathbb{N}$ with $n \neq 2$, $m \neq 2$ and $(n, m) \neq (3,4)$

$$\Sigma_{n,m}(R) \subsetneq \mathcal{P}_{n,m}(R).$$

2. Show that $\forall\, n \in \mathbb{N}$ and $\forall\, \alpha_1, \ldots, \alpha_n, x_1, \ldots, x_n \in \mathbb{R}^{\geqslant 0} = \{y \in \mathbb{R} : y \geqslant\ 0\}$

$$\sum_{i=1}^{n} \alpha_i = 1 \;\Rightarrow\; \alpha_1 x_1 + \cdots + \alpha_n x_n - x_1^{\alpha_1} \cdots x_n^{\alpha_n} \geqslant 0.$$

3. Consider the ternary sextic form

$$S(\mathrm{x},\mathrm{y},\mathrm{z}) = \mathrm{x}^4\mathrm{y}^2 + \mathrm{y}^4\mathrm{z}^2 + \mathrm{z}^4\mathrm{x}^2 - 3\mathrm{x}^2\mathrm{y}^2\mathrm{z}^2,$$

and the Motzkin's form

$$M(\mathrm{x},\mathrm{y},\mathrm{z}) = \mathrm{z}^6 + \mathrm{x}^4\mathrm{y}^2 + \mathrm{x}^2\mathrm{y}^4 - 3\mathrm{x}^2\mathrm{y}^2\mathrm{z}^2.$$

We established that $M(\mathrm{x},\mathrm{y},\mathrm{z})$ is psd but is not sos (See ÜB 6 Vorlesung RAG WS 2009/2010).

(*a*) Show that $S(\mathrm{x},\mathrm{y},\mathrm{z})$ is psd.

(*b*) Show that $S(\mathrm{x},\mathrm{y},\mathrm{z})$ is not sos considering the possible monomials in a representation of $S(\mathrm{x},\mathrm{y},\mathrm{z})$ as sos.

(*c*) Can you prove that $S(\mathrm{x},\mathrm{y},\mathrm{z})$ and $M(\mathrm{x},\mathrm{y},\mathrm{z})$ are not sos using Robinson's method? Justify your answer.