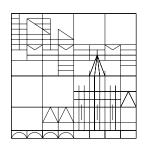
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Real Algebraic Geometry II Ordered Sets and Ordinal Numbers

1 Notation. Let A and B be sets. We denote by $A \sqcup B$ the disjoint union of A and B.

2 Definition. Let (A, \leq_A) and (B, \leq_B) be two ordered sets.

(a) We define the sum of ordered sets

$$(A, \leq_A) + (B, \leq_B) = A + B := (A \sqcup B, \leq_+)$$

such that for any $x_1, x_2 \in A \sqcup B$

$$x_1 \leq_+ x_2 \Leftrightarrow \begin{cases} x_1, x_2 \in A \text{ and } x_1 \leq_A x_2 & \text{or} \\ x_1, x_2 \in B \text{ and } x_1 \leq_B x_2 & \text{or} \\ x_1 \in A \text{ and } x_2 \in B & . \end{cases}$$

(b) We define the product of ordered sets

$$(A, \leq_A) . (B, \leq_B) = A.B := (A \times B, \leq_{\mathsf{rlex}})$$

such that $\leq_{\rm rlex}$ is the reverse lexicographic order, i.e. for any $(x_1,y_1), (x_2,y_2) \in A \times B$

$$(x_1, y_1) \leq_{\mathsf{rlex}} + (x_2, y_2) \Leftrightarrow \begin{cases} y_1 <_A y_2 & \text{or} \\ y_1 = y_2 \text{ and } x_1 \leq_B x_2 \end{cases}$$

3 Definition. A set A is called transitive if any element of A is also a subset of A.

4 Definition. A set α is called an ordinal if it is transitive and if (α, \in) is a well-ordered set.

5 Definition. Let (A, <) be a well-ordered set. The order type of (A, <), denoted ot(A), is defined as the unique ordinal to which (A, <) is isomorphic.

6 Definition. Let α and β be two ordinal numbers. Let $n \in \mathbb{N}$.

- (a) We define the sum of ordinals as $\alpha + \beta := \operatorname{ot} (\alpha + \beta)$.
- (b) We define $\omega.n := \underbrace{\omega + \cdots + \omega}_{n-\text{times}}$.
- (c) We define the product of ordinals $\alpha.\beta := \operatorname{ot}(\alpha.\beta)$.
- (d) We define $\omega^n := \underbrace{\omega. \cdots . \omega}_{n-\text{times}}$.