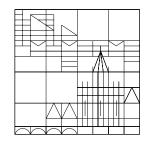
Universität Konstanz

Fachbereich Mathematik und Statistik

Prof. Dr. Salma Kuhlmann Katharina Dupont Gabriel Lehéricy



Real Algebraic Geometry II Exercise Sheet 6

Exercise 1 (archimedean equivalence)

Let G be an ordered abelian group. Let \sim^+ and \ll^+ on G be as defined in the lecture.

Show that

- (a) \sim^+ is an equivalence relation.
- (b) for all $x, y, z \in G \setminus \{0\}$ if $x \ll^+ y$ and $x \sim^+ z$ then $z \ll^+ y$.
- (c) for all $x, y, z \in G \setminus \{0\}$ if $x \ll^+ y$ and $y \sim^+ z$ then $x \ll^+ z$.

Exercise 2

Let G be an ordered abelian group. Let \sim^+ and \ll^+ on G be as defined in the lecture. Let $\Gamma := G/\sim^+=\{[x] \mid x \in G\setminus\{0\}\}$. Define a relation $<_{\Gamma}$ on Γ by

$$[y] <_{\Gamma} [x] \Leftrightarrow x \ll^+ y$$

for all $x, y \in G \setminus \{0\}$.

- (a) Show that Γ is a totally ordered set under $<_{\Gamma}$.
- (b) Define $v: G \longrightarrow \Gamma \cup \{\infty\}$ by $v(0) = \infty$ and v(x) = [x] for all $x \in G \setminus \{0\}$. Show that v is a valuation on G as a \mathbb{Z} -module.
- (c) Show that for all $y \in G$, if $\operatorname{sgn} x = \operatorname{sgn} y$, then $v(x+y) = \min\{v(x), v(y)\}$.
- (d) Let $x \in G \setminus \{0\}$. Show that $C_x = G^{v(x)}$ where $C_x := \bigcap \{C \mid C \text{ is a convex subgroup of } G \text{ and } x \in C\}.$
- (e) Let $x \in G \setminus \{0\}$. Show that $D_x = G_{v(x)}$ where $D_x := \bigcup \{C \mid C \text{ is a convex subgroup of } G \text{ and } x \notin D\}.$

(f) Conclude that B_x , the archimedean component of x in G, is equal to $B\left(G,v(x)\right)$, the homogeneous component corresponding to v(x), for all $x\in G\setminus\{0\}$.

1 Definition. Let C, D be convex subgroups of G with $C \subseteq D$. We call the pair (C, D) a jump, if whenever D' is a convex subgroup of G with $C \subseteq D' \subseteq D$ then D' = C or D' = D.

Exercise 3

Let $(G, <_G)$ be an ordered abelian group and C a convex subgroup of G. Let B := G/C.

- (a) We define on B a binary relation $<_B$ as follows: For all $g_1,g_2\in G$ let $g_1+C<_B$ g_2+C if and only if $g_1<_G$ g_2 and $g_1-g_2\notin C$. Show that $(B,<_B)$ is an totally ordered group.
- (b) Show that there is a bijective correspondence between the convex subgroups of B and the convex subgroups D of G with $C \subseteq D \subseteq G$.
- (c) Show that a totally ordered abelian group G is archimedean if and only if G and $\{0\}$ are its only convex subgroups.
- (d) Let D be a convex subgroup of G such that $C \subseteq D$. Conclude that if (C, D) is a jump, then D/C is archimedean.

The exercise will be collected **Thursday**, 28/05/2015 until 10.00 at box 13 near F 441.