Universität Konstanz

Fachbereich Mathematik und Statistik

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Real Algebraic Geometry II
Exercise Sheet 6

Exercise 1 (archimedean equivalence)
Let G be an ordered abelian group. Let $\sim^+$ and $\ll^+$ on G be as defined in the lecture.
Show that

(a) $\sim^+$ is an equivalence relation.

(b) for all $x, y, z \in G \setminus \{0\}$ if $x \ll^+ y$ and $x \sim^+ z$ then $z \ll^+ y$.

(c) for all $x, y, z \in G \setminus \{0\}$ if $x \ll^+ y$ and $y \sim^+ z$ then $x \ll^+ z$.

Exercise 2
Let G be an ordered abelian group. Let $\sim^+$ and $\ll^+$ on G be as defined in the lecture. Let $\Gamma := G/\sim^+ = \{[x] \mid x \in G \setminus \{0\}\}$. Define a relation $<_\Gamma$ on Γ by

$$[y] <_\Gamma [x] \Leftrightarrow x \ll^+ y$$

for all $x, y \in G \setminus \{0\}$.

(a) Show that Γ is a totally ordered set under $<_\Gamma$.

(b) Define $v : G \longrightarrow \Gamma \cup \{\infty\}$ by $v(0) = \infty$ and $v(x) = [x]$ for all $x \in G \setminus \{0\}$.
Show that v is a valuation on G as a $\mathbb{Z}$-module.

(c) Show that for all $, y \in G$, if sgn $x =$ sgn y, then $v(x+y) = \min\{v(x), v(y)\}$.

(d) Let $x \in G \setminus \{0\}$. Show that $C_x = G^{v(x)}$ where
$C_x := \bigcap \{C \mid C$ is a convex subgroup of G and $x \in C\}$.

(e) Let $x \in G \setminus \{0\}$. Show that $D_x = G_{v(x)}$ where
$D_x := \bigcup \{C \mid C$ is a convex subgroup of G and $x \notin D\}$.

(f) Conclude that B_x, the archimedean component of x in G, is equal to $B(G, v(x))$, the homogeneous component corresponding to $v(x)$, for all $x \in G \setminus \{0\}$.

1 Definition. Let C, D be convex subgroups of G with $C \subseteq D$.
We call the pair (C, D) a jump, if whenever D' is a convex subgroup of G with $C \subseteq D' \subseteq D$ then $D' = C$ or $D' = D$.

Exercise 3
Let $(G, <_G)$ be an ordered abelian group and C a convex subgroup of G. Let $B := G/C$.

(a) We define on B a binary relation $<_B$ as follows:
For all $g_1, g_2 \in G$ let $g_1 + C <_B g_2 + C$ if and only if $g_1 <_G g_2$ and $g_1 - g_2 \notin C$.

Show that $(B, <_B)$ is an totally ordered group.

(b) Show that there is a bijective correspondence between the convex subgroups of B and the convex subgroups D of G with $C \subseteq D \subseteq G$.

(c) Show that a totally ordered abelian group G is archimedean if and only if G and $\{0\}$ are its only convex subgroups.

(d) Let D be a convex subgroup of G such that $C \subseteq D$.

Conclude that if (C, D) is a jump, then D/C is archimedean.

The exercise will be collected **Thursday, 28/05/2015** until 10.00 at box 13 near F 441.
