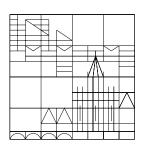
Universität Konstanz

Fachbereich Mathematik und Statistik

Prof. Dr. Salma Kuhlmann Katharina Dupont Gabriel Lehéricy



Real Algebraic Geometry II Exercise Sheet 8

Exercise 1

Let H be a hardy field. Define the asymptotic equivalence as in the lecture and denote the equivalence class of $f \in H$ by v(f) and the set of all equivalence classes by v(H). Let

$$\begin{array}{rcl} w:H & \longrightarrow & v(H) \cup \{\infty\} \\ f & \mapsto & \begin{cases} v(f) & \text{if } f \neq 0 \\ \infty & \text{if } f = 0 \end{cases}. \end{array}$$

- (a) Verify that asymptotic equivalence is an equivalence relation.
- (b) Let addition and order on v(H) be as defined in the lecture. Verify that v(H) is an ordered abelian group.
- (c) Show that w is a valuation on H.
- (d) Show that

$$H_{v} = \left\{ f \mid \lim_{x \to \infty} f(x) \in \mathbb{R} \right\}$$
$$I_{v} = \left\{ f \mid \lim_{x \to \infty} f(x) = 0 \right\}$$
$$\mathcal{U}_{v} = \left\{ f \mid \lim_{x \to \infty} f(x) \setminus \{0\} \in \mathbb{R} \right\}$$

(e) Show that w is equivalent to the natural valuation on H.

Definition: Let K be a field and let $v_1 : K \longrightarrow \Gamma_1 \cup \{\infty\}$ and $v_2 : K \longrightarrow \Gamma_2 \cup \{\infty\}$ be two valuations on K. v_1 and v_2 are called equivalent if $\mathcal{O}_{v_1} = \mathcal{O}_{v_2}$.

Exercise 2

Let $k \subseteq \mathbb{R}$ be a field and G an ordered abelian group. Let $\mathbb{K} = k((G))$ be the field of generalized power series endowed with the lexicographic order and the valuation $v := v_{\min}$.

- (a) Verify that $v(\mathbb{K}^{\times}) \cong G$.
- (b) Let $s = \sum_{g \in G} s(g)t^g \in \mathcal{O}_v$. Show that for the residue \overline{s} of s we have $\overline{s} = s(v(s))$.
- (c) Conclude that the residue field $\overline{\mathbb{K}}$ of \mathbb{K} is isomorphic to k.
- (d) Give an example of a field \mathbb{K} as above which is not real closed.
- (e) Show that if G is 2-divisible and k is square root closed for positive elements, then K is square root closed for positive elements.

Hint: Use Neumann's Lemma and the following identity (without proof):

$$(1+\varepsilon)^{\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \varepsilon^n$$

for $\alpha \in \mathbb{Q}^{>0}$ and $(\alpha)_n := \alpha \cdot (\alpha - 1) \cdots (\alpha - n + 1)$.

Then prove that for $\varepsilon \in \mathbb{K}$ with $v(\varepsilon) > 0$ we have $(1 + \epsilon)^{\frac{1}{2}} = \sqrt{1 + \varepsilon}$.

Definition:

- (a) Let G be an abelian group and $p \in \mathbb{N}$ prime. G is called p divisible if for every $g \in G$ there exists $h \in G$ such that $g = p \cdot h$.
- (b) Let K be an ordered field. K is called square root closed for positive elements if for every $x \in K^{>0}$ there exists $y \in K$ such that $x = y^2$.

The exercise will be collected **Thursday**, 11/06/2015 until 10.00 at box 13 near F 441.

http://www.math.uni-konstanz.de/~ dupont/rag.htm