Universität Konstanz

Fachbereich
Mathematik und Statistik

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Real Algebraic Geometry II
Exercise Sheet 8

Exercise 1
Let H be a hardy field. Define the asymptotic equivalence as in the lecture and denote the equivalence class of $f \in H$ by $v(f)$ and the set of all equivalence classes by $v(H)$.
Let

$$\begin{aligned} w : H &\longrightarrow v(H) \cup \{\infty\} \\ f &\mapsto \begin{cases} v(f) & \text{if } f \neq 0 \\ \infty & \text{if } f = 0 \end{cases}. \end{aligned}$$

(a) Verify that asymptotic equivalence is an equivalence relation.

(b) Let addition and order on $v(H)$ be as defined in the lecture. Verify that $v(H)$ is an ordered abelian group.

(c) Show that w is a valuation on H.

(d) Show that

$$\begin{aligned} H_v &= \left\{ f \mid \lim_{x\to\infty} f(x) \in \mathbb{R} \right\} \\ I_v &= \left\{ f \mid \lim_{x\to\infty} f(x) = 0 \right\} \\ \mathcal{U}_v &= \left\{ f \mid \lim_{x\to\infty} f(x) \setminus \{0\} \in \mathbb{R} \right\}. \end{aligned}$$

(e) Show that w is equivalent to the natural valuation on H.

Definition: Let K be a field and let $v_1 : K \longrightarrow \Gamma_1 \cup \{\infty\}$ and $v_2 : K \longrightarrow \Gamma_2 \cup \{\infty\}$ be two valuations on K.
v_1 and v_2 are called equivalent if $\mathcal{O}_{v_1} = \mathcal{O}_{v_2}$.

Exercise 2

Let $k \subseteq \mathbb{R}$ be a field and G an ordered abelian group. Let $\mathbb{K} = k((G))$ be the field of generalized power series endowed with the lexicographic order and the valuation $v := v_{\min}$.

(a) Verify that $v(\mathbb{K}^\times) \cong G$.

(b) Let $s = \sum_{g \in G} s(g)t^g \in \mathcal{O}_v$. Show that for the residue $\overline{s}$ of s we have $\overline{s} = s\big(v(s)\big)$.

(c) Conclude that the residue field $\overline{\mathbb{K}}$ of $\mathbb{K}$ is isomorphic to k.

(d) Give an example of a field $\mathbb{K}$ as above which is not real closed.

(e) Show that if G is 2-divisible and k is square root closed for positive elements, then $\mathbb{K}$ is square root closed for positive elements.

Hint: Use Neumann's Lemma and the following identity (without proof):

$$(1+\varepsilon)^\alpha = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!}\varepsilon^n$$

for $\alpha \in \mathbb{Q}^{>0}$ and $(\alpha)_n := \alpha \cdot (\alpha - 1) \cdots (\alpha - n + 1)$.

Then prove that for $\varepsilon \in \mathbb{K}$ with $v(\varepsilon) > 0$ we have $(1+\epsilon)^{\frac{1}{2}} = \sqrt{1+\varepsilon}$.

Definition:

(a) Let G be an abelian group and $p \in \mathbb{N}$ prime. G is called p divisible if for every $g \in G$ there exists $h \in G$ such that $g = p \cdot h$.

(b) Let K be an ordered field. K is called square root closed for positive elements if for every $x \in K^{>0}$ there exists $y \in K$ such that $x = y^2$.

The exercise will be collected **Thursday, 11/06/2015** until 10.00 at box 13 near F 441.
