Irreducibility and Rational Points
Problem Set 1

French-German Summer School
Galois Theory and Number Theory
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1. Let $g(X) \in \mathbb{Z}[X]$ be a polynomial. Assume that $g(n)$ is a perfect square for every positive integer $n \geq 1$.
   Prove (without using Hilbert's irreducibility theorem) that there exists $h(X) \in \mathbb{Z}[X]$ with $g = h^2$.

2. Let $f(T_1, \ldots, T_r, X) \in K[T, X] \smallsetminus K[T]$ (where $T = (T_1, \ldots, T_r)$) be a monic, irreducible, and Galois in X. For $\tau \in K^r$, we saw in class the implication:
   $$f(\tau, X) \quad \text{is irreducible} \quad \Longrightarrow \quad \mathrm{Gal}(f(\tau, X)/\mathbb{Q}) \cong \mathrm{Gal}(f(T, X)/K(T)).$$
   Show that the assumption "Galois" can't be dropped.

3. Show that the asymptotic density of a Hilbert set in the integers is 1; that is, if $H = H(f)$, then there exists $0 < \delta < 1$ such that
   $$\#\{1 \leq n \leq x : n \in H(f)\} = x + O(x^{1-\delta}).$$

4. Show that an Hilbert set H in $\mathbb{Q}$ is dense in the p-adic topology. (That is to say H intersect any open set $\{x \in \mathbb{Q} : v_p(x - a) \geq r\}$ for any $r > 0$ and $a \in \mathbb{Q}$.)

5. Show that $\mathbb{Q}_p$ is not Hilbertian.