Irreducibility and Rational Points – Lecture 1
Specializing Polynomials, Extensions and Covers

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Irreducibility and Rational Points

Lecture 1. Specializing polynomials, extensions and covers

Lecture 2. Rational points

Lecture 3. Specializing elliptic curves

Lecture 4. Varieties of Hilbert type

4. Examples and permanence principles

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6. $K = \mathbb{Q}^{\mathrm{tr}}$, the maximal totally real Galois extension of $\mathbb{Q}$

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Proof.

For $K_0 = \overline{K}_0$, this follows from Bertini's theorem:
For $f \in K_0[A, T, X]$ irreducible consider the variety

$$V = \{f = 0\} \subseteq \mathbb{A}^3$$

and the projection $\pi : \mathbb{A}^3 \to \mathbb{A}^2$. Then for a general hyperplane

$$H = \{aA + bT + c = 0\} \subseteq \mathbb{A}^2,$$

$$\pi|_V^{-1}(H) = \{f(A, \frac{aA + c}{-b}, X) = 0\}$$

is irreducible. □

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Example

$K_0((X,Y))$, $\mathrm{Quot}(\mathbb{Z}[[X]])$ is Hilbertian (Weissauer 1980)

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Let $f \in L[T, X]$ irreducible.
Idea for $L|K$ Galois and f^σ, $\sigma \in \mathrm{Gal}(L|K)$, distinct:

$$\tilde{f} := \prod_\sigma f^\sigma \in K[T, X]$$

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Now if $\tau \in K$ with $\tilde{f}(\tau, X)$ irreducible, then also $f(\tau, X)$ irreducible, as seen by unique factorization in $L[X]$. □

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2. ex. $K \subseteq L_0 \subseteq L$, $L_0|K$ Galois, $1 < [L : L_0] < \infty$ (Weissauer 1980), e.g. $L = \mathbb{Q}^{\mathrm{tr}}(\sqrt{-1})$

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4. ex. $K = M_0 \subseteq \cdots \subseteq M_r \supseteq L$, $M_{i+1}|M_i$ Galois, $\mathrm{Gal}(M_{i+1}|M_i)$ abelian or product of finite simple groups (Bary-Soroker–F.–Wiese 2015),
   e.g. $L \subseteq \mathbb{Q}^{[d]}$, the compositum of all degree $\leq d$ extensions

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5. $L \subseteq K(E_{\mathrm{tor}})$ for an elliptic curve $E|K$ ($\rightarrow$ Lecture 2,3)