Time-Asymptotic Stability for First-Order Symmetric Hyperbolic Systems of Balance Laws in Dissipative Compressible Fluid Dynamics

Heinrich Freistühler

Abstract

This paper identifies a non-(or /iso-)thermal variant of Ruggeri's 1983 formulation of viscous heat-conductive fluid dynamics as a hyperbolic system of balance laws and shows that both the original model and this variant have (a) time-asymptotically stable equilibria and (b) principal parts deriving from a protopotential: a single scalar function that induces the temporospatial flux as an appropriate part of its Hessian.

1 Ruggeri's first model and a nonthermal variant

The first appealing formulation for the dynamics of viscous heat-conductive fluids as a symmetric hyperbolic system of balance laws was given by Ruggeri in his groundbreaking 1983 paper [18]. It is of the form

$$\frac{\partial}{\partial t}\left(\frac{\partial X^0(\Upsilon)}{\partial \Upsilon}\right)+\sum_{j=1}^{3}\frac{\partial}{\partial x^j}\left(\frac{\partial X^j(\Upsilon)}{\partial \Upsilon}\right)=I(\Upsilon), \tag{1.1}$$

equations that are defined by given functions X^0, X^1, X^2, X^3, and I of the local state Υ of the fluid and to be solved for Υ as a function of time and space variables t, x_1, x_2, x_3. The form of the differential operator suggested in previous considerations by Godunov [10] and Boillat [3], Ruggeri chose the state variable ('main field') as

$$\Upsilon=(\tilde\psi,\tilde u,\tilde\theta,\tilde\Sigma,\tilde\sigma,\tilde q) \tag{1.2}$$

with

$$\tilde\psi\equiv\psi-\frac{u^2/2}{\theta},\ \tilde u\equiv\frac{u}{\theta},\ \tilde\theta=-\frac{1}{\theta},\ \tilde\Sigma\equiv\frac{\Sigma}{\theta},\ \tilde\sigma\equiv\frac{\sigma}{\theta},\ \tilde q\equiv\frac{q}{\theta^2}, \tag{1.3}$$

the potentials as

$$X^0 = \frac{p}{\theta}, \quad X^j = ((p\mathbb{I} + \theta(\tilde{\Sigma} + \tilde{\sigma}\mathbb{I}))\tilde{u})^j + \theta\tilde{q}^j \tag{1.4}$$

with

$$p = \hat{p}\left(\theta, \psi + \Psi(\tilde{\Sigma}, \tilde{\sigma}.\tilde{q})\right), \tag{1.5}$$

and the source as

$$I = \left(0_1, 0_3, 0_1, -\frac{\Sigma}{2\eta}, -\frac{\sigma}{\zeta}, -\frac{q}{\chi}\right), \tag{1.6}$$

where $u \in \mathbb{R}^3, \theta, \psi = g/\theta, -(\Sigma + \sigma\mathbb{I})$ with $\Sigma \in \mathbb{R}^{3\times 3}$ tracefree and symmetric, and $q \in \mathbb{R}^3$ denote velocity, temperature, thermal potential (with g the free enthalpy), viscous stress, and heat flux. In Ruggeri's model the fluid is specified by the pressure function $\hat{p}$, the positive dissipation coefficients η, ζ of viscosity, χ of thermal conductivity, and the associated 'extension' Ψ. Incorporating ideas of Maxwell [14] and Cattaneo [4] in an ingenious way, his equations (1.1)–(1.6) are a delayed version of the Navier-Stokes-Fourier system, to which they reduce when one replaces Ψ by 0.

His paper was a response to the ideas of "Extended Irreversible Thermodynamics" (cf. references in [18], p. 169) that had developed starting with Müller's early paper [15]. In their ensuing extensive work leading to the fundamental theory of *Rational Extended Thermodynamics* (RET), Ruggeri and Müller proceeded to more refined formulations of dissipative compressible fluid dynamics. Still first-order symmetric hyperbolic systems of balance laws, these are based on main fields that are different from (1.3), and there is an infinite hierarchy of such formulations, both in the classical and in the relativistic setting [16].

Despite these later refinements, it seems that the abovedescribed 'Ruggeri's first model' still deserves attention at least from mathematical points of view, as it may serve as a prototype regarding the latter. That is why in this note we return to it and do three things: We (i) identify a *nonthermal version* thereof, (ii) establish the *time-asymptotic stability* of homogeneous equilibrium solutions for both this nonthermal variant and the original model, and (iii) formulate the existence of a *protopotential* as a natural requirement on symmetric hyperbolic systems of balance laws modelling the dynamics of compressible fluids.

Reconsidering Ruggeri's model for an isothermal situation suggests a nonthermal variant that uses

$$\Upsilon = (\tilde{h}, u, \Sigma, \sigma) \quad \text{with} \quad \tilde{h} \equiv h - u^2/2, \tag{1.7}$$

as state variable, where u, Σ, σ are as before and h denotes enthalpy. This variant is again of the form (1.1), with now

$$X^0 = p, \quad X^j = ((p\mathbb{I} + (\Sigma + \sigma\mathbb{I}))\, u)^j\,, \tag{1.8}$$

where

$$p = \hat{p}(h + H(\Sigma, \sigma)), \tag{1.9}$$

and

$$I = \left(0_1, 0_3, -\frac{\Sigma}{2\eta}, -\frac{\sigma}{\zeta}\right). \tag{1.10}$$

The formulation (1.1),(1.7)-(1.10) is similar to one proposed by Yong [22].

Before turning to the stability question (in Secs. 2 and 3) and the protopotential (in Sec. 4), we now state our assumptions on the 'non-extended' equations of state $p = \hat{p}(h)$ and $p = \hat{p}(\theta, \psi)$. They are

$$\hat{p}'(h) > 0 \text{ and } \hat{p}''(h) > 0, \tag{1.11}$$

resp.

$$\hat{p}_\psi(\theta, \psi) > 0 \text{ and } \begin{pmatrix} \theta^{-1}\hat{p}_{\psi\psi}(\theta, \psi) & (-\hat{p}_\psi + \theta\hat{p}_{\theta\psi})(\theta, \psi) \\ (-\hat{p}_\psi + \theta\hat{p}_{\theta\psi})(\theta, \psi) & \theta^3\hat{p}_{\theta\theta}(\theta, \psi) \end{pmatrix} > 0. \tag{1.12}$$

In either case, the first condition just states positivity of the material density ρ which in the non-extended context equals $\hat{p}'(h)$ or $\theta^{-1}\hat{p}_\psi(\theta, \psi)$, respectively, while the second condition means convexity of the pressure as a function of the enthalpy h or as a function $\tilde{p}(g, \theta) = \hat{p}(\theta, g/\theta)$ of temperature θ and free enthalpy g, respectively. To see the latter, one confirms by a straightforward calculation that $(1.12)_2$ is equivalent to

$$\begin{pmatrix} \tilde{p}_{gg}(g, \theta) & \tilde{p}_{g\theta}(g, \theta) \\ \tilde{p}_{\theta g}(g, \theta) & \tilde{p}_{\theta\theta}(g, \theta) \end{pmatrix} > 0. \tag{1.13}$$

Condition (1.13) (see also [20]) is the same as requiring convexity of the specific energy as a function $e(v, s)$ of specific volume v and specific entropy s,

$$\begin{pmatrix} e_{vv}(v, s) & e_{vs}(v, s) \\ e_{sv}(v, s) & e_{ss}(v, s) \end{pmatrix} > 0, \tag{1.14}$$

as is obvious from the fact that $g(p, \theta)$ and $e(v, s)$ are Legendre conjugate; similarly, in the nonthermal case where $e = e(v)$ is the Legendre transform of $h = h(p)$, $(1.11)_2$ is equivalent to

$$e''(v) > 0. \tag{1.15}$$

Assumptions (1.12),(1.11) thus just reexpress the well-known standard requirements on the fluid's equation of state that make its usual Euler equations hyperbolic.

The considerations of the following sections hold whenever the extensions H or Ψ of the (free) enthalpy are general strictly convex functions of $\tilde{\Sigma}, \tilde{\sigma}$ (/ and $\tilde{q}$) that attain their minimal value 0 at the origin. For the sake of concreteness, we do however also write down the equations of motion for an "interesting special case" highlighted already in Sec. 7 of [18] and an analogous case in the nonthermal setting; see (2.6) and (3.2).

My thinking leading to this paper (cf. [6] for documentation of its earlier steps) was prompted by Yong's pioneering paper [22]; independently of [18], Yong also had the fine idea of revising Maxwell's constitutive relation

$$\epsilon\dot{\Sigma} + \Sigma = -\eta\left[\nabla u + (\nabla u)^\top - \frac{2}{3}\nabla\cdot v\right] - \zeta\nabla\cdot uI$$

by relaxing different dissipative effects, here shear viscosity and bulk viscosity, separately.

2 Stability for Ruggeri's first model

The purpose of this section is to show the following.

Theorem 1. *Consider system* (1.1)–(1.6), *with $\hat{p}$ satisfying the standard conditions* (1.12), *Ψ assuming its minimum at* 0 *and strictly convex in the sense that its Hessian $D^2\Psi$ is positive definite. Let $\theta_*, \psi_* \in \mathbb{R}$ be physically meaningful values of the temperature and the thermal potential (in particular, θ_* must be positive), and Υ_0 regular*[1] *data on $\mathbb{R}^3$ for which the H^s norm, with $s > 5/2$, of $\Upsilon_0 - (\psi_*, 0, \theta_*, 0, 0, 0)$ is sufficiently small. Then the unique global solution Υ with these data, $\Upsilon(0, .) = \Upsilon_0$, decays as*

$$\|D_x^\beta(\Upsilon(t, .) - \Upsilon_0)\|_{L^2(\mathbb{R}^3)} \sim t^{-\frac{3}{4}-\frac{|\beta|}{2}} \quad \textit{for } t \to \infty. \tag{2.1}$$

Very much like Ruggeri did for his own analogous observation on a more elaborate model of relativistic fluid dynamics in Sec. 6.2 of [17], we show this by appealing to a deep result of Bianchini, Hanouzet, Natalini on general 'partially dissipative hyperbolic systems with a convex entropy' [2].

By Theorem 5.4 in [2], Theorem 1 is a direct consequence of the following observation on the Hessians of the Godunov-Boillat potentials X^0, X^1, X^2, X^3 from (1.4) with respect to the main field (1.3).

Proposition 1. *The Kawashima condition is satisfied:*

$$\textit{No eigenvector of } \xi_j D^2X^j, \xi \neq 0, \textit{ with respect to } D^2X^0 \textit{ lies in } ker(DI). \tag{2.2}$$

Condition (2.2), originally first formulated for hyperbolic-parabolic systems, means that the single equations of (1.1) are coupled in a way that transfers the damping effect of the terms present on the right-hand sides of some of the equations to all of them [12, 19, 21].

The rest of this section is devoted to proving Proposition 1.

[1] In order to get (2.1) for all multiindices up to any given order, the data must have sufficiently many derivatives in L^2. Cf. Sec. 5 of [2].

The first derivatives of the potentials with respect to the main field variables are

$$
\begin{aligned}
\frac{\partial X^0}{\partial \tilde\psi} &= \frac{\hat p_\psi}{\theta} & \frac{\partial X^j}{\partial \tilde\psi} &= \hat p_\psi \tilde u^j \\
\frac{\partial X^0}{\partial \tilde u_m} &= \hat p_\psi \tilde u^m & \frac{\partial X^j}{\partial \tilde u_m} &= \hat p_\psi \theta \tilde u^m \tilde u^j + (\hat p + \theta\tilde\sigma)\delta^{mj} + \theta\tilde\Sigma^{mj} \\
\frac{\partial X^0}{\partial \tilde\theta} &= -\hat p + \theta \hat p_\theta + \frac12 \hat p_\psi \theta |\tilde u|^2 & \frac{\partial X^j}{\partial \tilde\theta} &= \theta^2 \left(\left(\left(\hat p_\theta + \frac12 \hat p_\psi \theta |\tilde u|^2 \right) \mathbb{I} + (\tilde\Sigma + \tilde\sigma \mathbb{I}) \right) \tilde u + \tilde q \right)^j \\
\frac{\partial X^0}{\partial \tilde\Sigma_{kl}} &= \frac{\hat p_\psi}{\theta} \frac{\partial \Psi}{\partial \tilde\Sigma_{kl}} & \frac{\partial X^j}{\partial \tilde\Sigma_{kl}} &= \theta \left(C^{ijkl} \tilde u_i + \frac{\hat p_\psi}{\theta} \frac{\partial \Psi}{\partial \tilde\Sigma_{kl}} \tilde u^j \right) \\
\frac{\partial X^0}{\partial \tilde\sigma} &= \frac{\hat p_\psi}{\theta} \frac{\partial \Psi}{\partial \tilde\sigma} & \frac{\partial X^j}{\partial \tilde\sigma} &= \theta \left(1 + \frac{\hat p_\psi}{\theta} \frac{\partial \Psi}{\partial \tilde\sigma} \right) \tilde u^j \\
\frac{\partial X^0}{\partial \tilde q_m} &= \frac{\hat p_\psi}{\theta} \frac{\partial \Psi}{\partial \tilde q_m} & \frac{\partial X^j}{\partial \tilde q_m} &= \theta \left(\delta^{mj} + \frac{\hat p_\psi}{\theta} \frac{\partial \Psi}{\partial \tilde q_m} \tilde u^j \right),
\end{aligned}
\tag{2.3}
$$

where, as henceforth, we use

$$
C^{ijkl} := \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} - \frac23 \delta^{kl}\delta^{ij} \tag{2.4}
$$

and the Einstein summation convention. Note that the tracefree symmetric character of Σ relates its nine components Σ_{kl} among each other; in differentiating the term $\tilde\Sigma\tilde u$ we have correspondingly treated it as as $\tilde\Sigma'\tilde u$ where $\tilde\Sigma' = \Sigma'/\theta$ with

$$
\Sigma' \equiv \begin{pmatrix} \frac23 \Sigma_{11} - \frac13 (\Sigma_{22} + \Sigma_{33}) & \frac12 (\Sigma_{12} + \Sigma_{21}) & \frac12 (\Sigma_{13} + \Sigma_{31}) \\ \frac12 (\Sigma_{21} + \Sigma_{12}) & \frac23 \Sigma_{22} - \frac13 (\Sigma_{11} + \Sigma_{33}) & \frac12 (\Sigma_{23} + \Sigma_{32}) \\ \frac12 (\Sigma_{31} + \Sigma_{13}) & \frac12 (\Sigma_{32} + \Sigma_{23}) & \frac23 \Sigma_{33} - \frac13 (\Sigma_{11} + \Sigma_{22}) \end{pmatrix}. \tag{2.5}
$$

With the extended density and extended eneryg $\rho = \hat p_\psi/\theta$ and $\rho e = \theta \hat p_\theta - \hat p$, one readily confirms that in the *interesting special case* that

$$
\Psi(\tilde\Sigma, \tilde\sigma.\tilde q) = \frac12 \tau_1 \tilde\Sigma : \tilde\Sigma + \frac12 \tau_2 \tilde\sigma^2 + \frac12 \tau_3 \tilde q^2,
$$

the equations of motion read (cf. [18], Secs. 5,7)

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \mathrm{div}_x(\rho u) &= 0 \\
\frac{\partial(\rho u)}{\partial t} + \mathrm{div}_x(\rho u \otimes u + (p+\sigma)\mathbb{I} + \Sigma) &= 0 \\
\frac{\partial(\rho(e + |u|^2/2))}{\partial t} + \mathrm{div}_x((\rho(e+|u|^2/2) + p + \sigma)\mathbb{I} + \Sigma)u) &= 0 \\
\tau_1 \left(\frac{\partial(\rho\tilde\Sigma)}{\partial t} + \mathrm{div}_x(\rho\tilde\Sigma \otimes u) \right) + \mathrm{div}_x(C \otimes u) &= (-1/2\eta)\Sigma \\
\tau_2 \left(\frac{\partial(\rho\tilde\sigma)}{\partial t} + \mathrm{div}_x(\rho\tilde\sigma u) \right) + \mathrm{div}_x u &= (-1/\zeta)\sigma \\
\tau_3 \left(\frac{\partial(\rho\tilde q)}{\partial t} + \mathrm{div}_x(\rho\tilde q \otimes u) \right) + \mathrm{grad}_x \theta &= (-1/\chi) q.
\end{aligned}
\tag{2.6}
$$

Lemma 1. *(i) The Hessian D^2X^0 is always positive definite.*
(ii) For $\tilde{u} = \tilde{\Sigma} = \sigma = \tilde{q} = 0$, the Hessians

$$D^2X^\alpha = \begin{pmatrix} X^\alpha_{\tilde\psi\tilde\psi} & X^\alpha_{\tilde\psi\tilde u_n} & X^\alpha_{\tilde\psi\tilde\theta} & X^\alpha_{\tilde\psi\tilde\Sigma_{rs}} & X^\alpha_{\tilde\psi\tilde\sigma} & X^\alpha_{\tilde\psi\tilde q_n} \\ X^\alpha_{\tilde u_m\tilde\psi} & X^\alpha_{\tilde u_m\tilde u_n} & X^\alpha_{\tilde u_m\tilde\theta} & X^\alpha_{\tilde u_m\tilde\Sigma_{rs}} & X^\alpha_{\tilde u_m\tilde\sigma} & X^\alpha_{\tilde u_m\tilde q_n} \\ X^\alpha_{\tilde\theta\tilde\psi} & X^\alpha_{\tilde\theta\tilde u_n} & X^\alpha_{\tilde\theta\tilde\theta} & X^\alpha_{\tilde\theta\tilde\Sigma_{rs}} & X^\alpha_{\tilde\theta\tilde\sigma} & X^\alpha_{\tilde\theta\tilde q_n} \\ X^\alpha_{\tilde\Sigma_{kl}\tilde\psi} & X^\alpha_{\tilde\Sigma_{kl}\tilde u_n} & X^\alpha_{\tilde\Sigma_{kl}\tilde\theta} & X^\alpha_{\tilde\Sigma_{kl}\tilde\Sigma_{rs}} & X^\alpha_{\tilde\Sigma_{kl}\tilde\sigma} & X^\alpha_{\tilde\Sigma_{kl}\tilde q_n} \\ X^\alpha_{\tilde\sigma\tilde\psi} & X^\alpha_{\tilde\sigma\tilde u_n} & X^\alpha_{\tilde\sigma\tilde\theta} & X^\alpha_{\tilde\sigma\tilde\Sigma_{rs}} & X^\alpha_{\tilde\sigma\tilde\sigma} & X^\alpha_{\tilde\sigma\tilde q_n} \\ X^\alpha_{\tilde q_m\tilde\psi} & X^\alpha_{\tilde q_m\tilde u_n} & X^\alpha_{\tilde q_m\tilde\theta} & X^\alpha_{\tilde q_m\tilde\Sigma_{rs}} & X^\alpha_{\tilde q_m\tilde\sigma} & X^\alpha_{\tilde q_m\tilde q_n} \end{pmatrix}, \quad \alpha = 0, 1, 2, 3,$$

assume the values

$$D^2X^0|_E = \begin{pmatrix} \theta^{-1}\hat p_{\psi\psi} & 0 & -\hat p_\psi + \theta\hat p_{\theta\psi} & 0 & 0 & 0 \\ 0 & \hat p_\psi & 0 & 0 & 0 & 0 \\ -\hat p_\psi + \theta\hat p_{\theta\psi} & 0 & \theta^3\hat p_{\theta\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Psi_{\tilde\Sigma_{kl}\tilde\Sigma_{rs}}(0) & \Psi_{\tilde\Sigma_{kl}\tilde\sigma}(0) & \Psi_{\tilde\Sigma_{kl}\tilde q_n}(0) \\ 0 & 0 & 0 & \Psi_{\tilde\sigma\tilde\Sigma_{rs}}(0) & \Psi_{\tilde\sigma\tilde\sigma}(0) & \Psi_{\tilde\sigma\tilde q_n}(0) \\ 0 & 0 & 0 & \Psi_{\tilde q_m\tilde\Sigma_{rs}}(0) & \Psi_{\tilde q_m\tilde\sigma}(0) & \Psi_{\tilde q_m\tilde q_n}(0) \end{pmatrix}$$

$$D^2X^j|_E = \begin{pmatrix} 0 & \hat p_\psi\delta^{nj} & 0 & 0 & 0 & 0 \\ \hat p_\psi\delta^{mj} & 0 & \theta^2\hat p_\theta\delta^{mj} & \theta C^{mjrs} & \theta\delta^{mj} & 0 \\ 0 & \theta^2\hat p_\theta\delta^{nj} & 0 & 0 & 0 & \theta^2\delta^{nj} \\ 0 & \theta C^{njkl} & 0 & 0 & 0 & 0 \\ 0 & \theta\delta^{nj} & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta^2\delta^{mj} & 0 & 0 & 0 \end{pmatrix}, \quad j = 1, 2, 3,$$

while the Jacobian of the source is

$$DI|_E = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1/2\eta)\delta^{(kl)(rs)} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/\zeta & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/\chi\delta^{mn} \end{pmatrix}.$$

Proof. (i) As $D^2\Psi > 0$ by assumption, transforming

$$D^2X^0 = -\tilde\theta\hat p_{\psi\psi}\begin{pmatrix} 1 & \theta\tilde u^\top & 0 & D\Psi \\ \theta\tilde u & \theta^2\tilde u\otimes\tilde u & 0 & \theta\tilde u\otimes D\Psi \\ 0 & 0 & 0 & 0 \\ D^\top\Psi & (\theta\tilde u\otimes D\Psi)^\top & 0 & (D\Psi)^\top D\Psi \end{pmatrix}$$
$$+\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \hat p_{\tilde\psi}I & \theta\hat p_{\tilde\psi}\tilde u^\top & 0 \\ 0 & \theta\hat p_{\tilde\psi}\tilde u & X^0_{\tilde\theta\tilde\theta} & 0 \\ 0 & 0 & 0 & \theta^{-1}\hat p_{\tilde\psi}D^2\Psi \end{pmatrix} + X^0_{\tilde\theta\tilde\psi}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \theta\tilde u^\top & 0 \\ 1 & \theta\tilde u & 0 & D\Psi \\ 0 & 0 & D\Psi & 0 \end{pmatrix}$$

with

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -D^\top\Psi & 0 & 0 & I \end{pmatrix}$$

shows that $M(D^2X^0)M^\top$

$$= -\tilde{\theta}\hat{p}_{\psi\psi}\begin{pmatrix} 1 & \theta\tilde{u}^\top & 0 & 0 \\ \theta\tilde{u} & \theta^2\tilde{u}\otimes\tilde{u} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \hat{p}_{\tilde{\psi}}I & \theta\hat{p}_{\tilde{\psi}}\tilde{u}^\top & 0 \\ 0 & \theta\hat{p}_{\tilde{\psi}}\tilde{u} & X^0_{\tilde{\theta}\tilde{\theta}} & 0 \\ 0 & 0 & 0 & \rho D^2\Psi \end{pmatrix} + X^0_{\tilde{\theta}\tilde{\psi}}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \theta\tilde{u}^\top & 0 \\ 1 & \theta\tilde{u} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} D^2_{\tilde{\psi},\tilde{u},\tilde{\theta}}(\hat{p}/\theta) & 0 \\ 0 & \rho D^2\Psi \end{pmatrix}.$$

and thus D^2X^0 itself are positive definite iff only

$$D^2_{\tilde{\psi},\tilde{u},\tilde{\theta}}(\hat{p}/\theta) > 0.$$

The latter is a well known consequence of the facts that (i) by virtue of (1.14), the entropy $S = \rho s$ is a strictly convex function of the conserved quantities $\rho, m = \rho u, \mathcal{E} = \rho(e + |u|^2/2)$, and (ii) X^0 is the Legendre transform of S, with $\tilde{\psi}, \tilde{u}, \tilde{\theta}$ as argument variables that are Legendre dual to $\rho, m, \mathcal{E}$ [10, 5].

(ii) now follows by straightforward computation. $\square$

Now, $\ker(DI)$ consists exactly of all vectors of the form $\bar{\Upsilon} = (\bar{\psi}, (\bar{u}_n)_n, \bar{\theta}, 0, 0, 0)^\top$ and if for any such v and any $\lambda \in \mathbb{R}, \xi \in \mathbb{R}^3 \setminus \{0\}$,

$$(-\lambda D^2X^0 + \xi_j D^2X^j)\bar{\Upsilon} = 0,$$

then necessarily $(\bar{u}_n)_n = 0, \bar{\theta} = 0$ and thus also $\bar{\psi} = 0$. Theorem 1 is proved.

Remark on hyperbolicity. Both in Lemma 1 and Lemma 2, the respective assertion (i) confirms that the system in question is symmetric hyperbolic in the sense of Friedrichs [8, 1] everywhere in its state space. Cf. Sec. 6 of [18].

Remark on possible dichotomy. Dynamical systems of PDE – cf., e. g., [13] regarding systems of balance laws – can of course be dichotomous in the sense that while sufficiently small perturbations of a homogeneous reference state induce global smooth solutions which time-asymptotically decay to that state, large data may lead to blowup in finite time. Hu et al. have recently demonstrated the latter for a model which is somewhat similar to Ruggeri's, but not completely in the form of balance laws [11]. Could their methods be used to show blowup of classical solutions to Ruggeri's model?

3 Stability for the nonthermal model

Here we demonstrate the counterpart of Theorem 1 for the nonthermal model:

Theorem 2. *Consider the nonthermal variant* (1.1),(1.7)-(1.10) *of Ruggeri's model, with* $\hat{p}$ *satisfying the standard conditions* (1.11)*,* H *assuming its minimum at* 0 *and strictly convex in the sense that its Hessian* D^2H *is positive definite. Let* $\psi_* \in \mathbb{R}$ *be a physically meaningful value of the thermal potential and* Υ_0 *regular*[1] *data on* $\mathbb{R}^3$ *for which the* H^s *norm, with* $s > 5/2$*, of* $\Upsilon_0 - (\psi_*, 0, 0, 0)$ *is sufficiently small. Then the unique global solution* Υ *with these data,* $\Upsilon(0, .) = \Upsilon_0$*, decays as* (2.1)*.*

By the same argumentation as in Sec. 2, this follows once we have shown that Proposition 1 holds also for the Hessians of the potentials X^0, X^1, X^2, X^3 from (1.8) with respect to the Godunov variables (1.7). This is what the rest of this section serves to.

The first derivatives are

$$
\begin{aligned}
\frac{\partial X^0}{\partial \tilde{h}} &= \rho & \frac{\partial X^j}{\partial \tilde{h}} &= \rho u^j \\
\frac{\partial X^0}{\partial u_m} &= \rho u^m & \frac{\partial X^j}{\partial u_m} &= \rho u^m u^j + (p+\sigma)\delta^{mj} + \Sigma^{mj} \\
\frac{\partial X^0}{\partial \Sigma_{kl}} &= \rho \frac{\partial H}{\partial \Sigma_{kl}} & \frac{\partial X^j}{\partial \Sigma_{kl}} &= C^{ijkl} u_i + \rho \frac{\partial H}{\partial \Sigma_{kl}} u^j \\
\frac{\partial X^0}{\partial \sigma} &= \rho \frac{\partial H}{\partial \sigma} & \frac{\partial X^j}{\partial \sigma} &= u^j + \rho \frac{\partial H}{\partial \sigma} u^j
\end{aligned}
\tag{3.1}
$$

with C^{ijkl} as in (2.4) and

$$\rho = \hat{p}'(h+H).$$

If, for instance,

$$H = \frac{1}{2}\tau_1 \Sigma : \Sigma + \frac{1}{2}\tau_2 \sigma^2,$$

the equations of motion thus read

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \mathrm{div}_x(\rho u) &= 0 \\
\frac{\partial \rho u}{\partial t} + \mathrm{div}_x(\rho u \otimes u + (p+\sigma)\mathbb{I} + \Sigma) &= 0 \\
\tau_1 \left(\frac{\rho \Sigma}{\partial t} + \mathrm{div}_x(\rho u \otimes \Sigma) \right) + \mathrm{div}_x(C \otimes u) &= (-1/2\eta)\Sigma \\
\tau_2 \left(\frac{\rho \sigma}{\partial t} + \mathrm{div}_x(\rho u \sigma) \right) + \mathrm{div}_x(u) &= (-1/\zeta)\sigma.
\end{aligned}
\tag{3.2}
$$

Lemma 2. *(i) The Hessian*

$$D^2X^0 = \begin{pmatrix} X^0_{\tilde{h}\tilde{h}} & X^0_{\tilde{h}u_n} & X^0_{\tilde{h}\Sigma_{rs}} & X^0_{\tilde{h}\sigma} \\ X^0_{u_m\tilde{h}} & X^0_{u_mu_n} & X^0_{u_m\Sigma_{rs}} & X^0_{u_m\sigma} \\ X^0_{\Sigma_{kl}\tilde{h}} & X^0_{\Sigma_{kl}u_n} & X^0_{\Sigma_{kl}\Sigma_{rs}} & X^0_{\Sigma_{kl}\sigma} \\ X^0_{\sigma\tilde{h}} & X^0_{\sigma u_n} & X^0_{\sigma\Sigma_{rs}} & X^0_{\sigma\sigma} \end{pmatrix}$$

of the temporal potential X^0 with respect to the main field is always positive definite.
(ii) For $u = \Sigma = \sigma = 0$, it assumes the form

$$D^2X^0|_E = \begin{pmatrix} \hat{p}'' & 0 & 0 & 0 \\ 0 & \rho\delta^{mn} & 0 & 0 \\ 0 & 0 & H_{\Sigma_{kl}\Sigma_{rs}}(0) & H_{\Sigma_{kl}\sigma})(0) \\ 0 & 0 & H_{\sigma\Sigma_{rs}}(0) & H_{\sigma\sigma}(0) \end{pmatrix}, \tag{3.3}$$

and the Hessians of the spatial potentials are

$$D^2X^j|_E = \begin{pmatrix} X^j_{\tilde{h}\tilde{h}} & X^j_{\tilde{h}u_n} & X^j_{\tilde{h}\Sigma_{rs}} & X^j_{\tilde{h}\sigma} \\ X^j_{u_m\tilde{h}} & X^j_{u_mu_n} & X^j_{u_m\Sigma_{rs}} & X^j_{u_m\sigma} \\ X^j_{\Sigma_{kl}\tilde{h}} & X^j_{\Sigma_{kl}u_n} & X^j_{\Sigma_{kl}\Sigma_{rs}} & X^j_{\Sigma_{kl}\sigma} \\ X^j_{\sigma\tilde{h}} & X^j_{\sigma u_n} & X^j_{\sigma\Sigma_{rs}} & X^j_{\sigma\sigma} \end{pmatrix} = \begin{pmatrix} 0 & \rho\delta^{nj} & 0 & 0 \\ \rho\delta^{mj} & 0 & C^{mjrs} & \delta^{mj} \\ 0 & C^{njkl} & 0 & 0 \\ 0 & \delta^{nj} & 0 & 0 \end{pmatrix},$$

while the Jacobian of the source is

$$DI|_E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(1/2\eta)\delta^{(kl)(rs)} & 0 \\ 0 & 0 & 0 & -1/\zeta \end{pmatrix}.$$

Proof. (i) We readily find

$$DX^0 = \hat{p}'(.)\left(1, u^\top, DH\right)$$

and

$$D^2X^0 = \hat{p}'(.)\begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & D^2H \end{pmatrix} + \hat{p}''(.)\left(1, u^\top, DH\right)^\top \left(1, u^\top, DH\right),$$

which is always positive by virtue of (1.11) and obviously reduces to (3.3) at rest and equilibrium.

(ii) results from a straightforward computation. $\square$

Now, $\ker(DI)$ consists exactly of all vectors of the form $\bar{\Upsilon} = (\bar{h}, (\bar{u}_n)_n, 0, 0)^\top$ and if for any such $\bar{\Upsilon}$ and any $\lambda \in \mathbb{R}, \xi \in \mathbb{R}^3 \setminus \{0\}$,

$$(-\lambda D^2X^0 + \xi_j D^2X^j)\bar{\Upsilon} = 0,$$

then necessarily $(\bar{u}_n)_n = 0$ and thus $\bar{h} = 0$. Theorem 2 is proved.

4 Protopotentials

Both models considered above are based on a Godunov-Boillat type differential operator

$$\frac{\partial}{\partial t}\left(\frac{\partial X^0(\Upsilon)}{\partial \Upsilon}\right)+\sum_{j=1}^{d}\frac{\partial}{\partial x^j}\left(\frac{\partial X^j(\Upsilon)}{\partial \Upsilon}\right) \tag{4.1}$$

where the values of $\Upsilon=(\Upsilon^0,...,\Upsilon^{n-1})$ range in some convex state space $U\subset\mathbb{R}^n$. For any such operator we will call a function $X:U\to\mathbb{R}$ a *protopotential* if $U=V\times W$ with $V\subset\mathbb{R}^4$ and

$$X^\alpha(\Upsilon)=\frac{\partial X(\Upsilon)}{\partial \Upsilon_\alpha},\quad \alpha=0,1,2,3. \tag{4.2}$$

The existence of a protopotential is equivalent to the field

$$T^{\alpha\beta}=\frac{\partial X^\alpha(\Upsilon)}{\partial \Upsilon_\beta} \tag{4.3}$$

being symmetric in $\alpha,\beta\in\{0,1,2,3\}$. A prototypical case of this was noticed by Geroch and Lindblom [9] in relativistic fluid dynamics, where $T^{\alpha\beta}$ is the energy-momentum-stress tensor.

In classical fluid dynamics, $T^{\alpha\beta}$ is the Galilei invariant 4×4 *mass-momentum-stress* tensor

$$\begin{pmatrix}\rho & \rho u^\top\\ \rho u & \rho u\otimes u+(\Sigma+(p+\sigma)\mathbb{I})\end{pmatrix}, \tag{4.4}$$

which is naturally symmetric already since it is the non-relativistic residue of the Lorentz invariant energy-momentum-stress tensor (cf., e. g., [7], p. 17). The two models considered above correspondingly possess protopotentials. As one easily checks, the function

$$X(\tilde h,u,\Sigma,\sigma)=\hat P(h+H(\Sigma,\sigma))+\frac{1}{2}u^\top(\Sigma+\sigma\mathbb{I})u \tag{4.5}$$

with $\hat P'=\hat p$ is a protopotential for the nonthermal model (1.1),(1.7)-(1.10) with respect to $(\Upsilon_0,\Upsilon_1,\Upsilon_2,\Upsilon_3)=(\tilde h,u,\Sigma,\sigma)$, and

$$X(\tilde\psi,\tilde u,\tilde\theta,\tilde\Sigma,\tilde\sigma,\tilde q)=\hat P(\theta,\psi+\Psi(\tilde\Sigma,\tilde\sigma,\tilde q))/\theta+\frac{1}{2}u\cdot(\Sigma+\sigma\mathbb{I})u+q\cdot u \tag{4.6}$$

with $\hat P_\psi=\hat p$ is a protopotential for Ruggeri's model (1.1)–(1.6) with respect to $(\Upsilon_0,\Upsilon_1,\Upsilon_2,\Upsilon_3)=(\tilde\psi,\tilde u,\tilde\Sigma,\tilde\sigma)$.

This structural property has helped me in finding the nonthermal model. It might also be a guideline in further attempts to identify appropriate formulations of fluid dynamics in terms of systems of balance laws.

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