Corollary 2.2.4. *For a t.v.s. X the following are equivalent:*
a) X is Hausdorff.
b) the intersection of all neighbourhoods of the origin o is just $\{o\}$.
c) $\{0\}$ is closed.

Note that in a t.v.s. $\{0\}$ is closed is equivalent to say that all singletons are closed (and so that the space is (T1)).

Proof.
a)⇒ b) Let X be a Hausdorff t.v.s. space. Clearly, $\{o\} \subseteq \cap_{U\in\mathcal{F}(o)}U$. Now if b) does not hold, then there exists $x \in \cap_{U\in\mathcal{F}(o)}U$ with $x \neq o$. But by the previous theorem we know that (2.1) holds and so there exists $V \in \mathcal{F}(o)$ s.t. $x \notin V$ and so $x \notin \cap_{U\in\mathcal{F}(o)}U$ which is a contradiction.

b)⇒ c) Assume that $\cap_{U\in\mathcal{F}(o)}U = \{o\}$. If $x \in \overline{\{o\}}$ then $\forall\, V_x \in \mathcal{F}(x)$ we have $V_x \cap \{o\} \neq \emptyset$, i.e. $o \in V_x$. By Corollary 2.1.9 we know that each $V_x = U + x$ with $U \in \mathcal{F}(o)$. Then $o = u + x$ for some $u \in U$ and so $x = -u \in -U$. This means that $x \in \cap_{U\in\mathcal{F}(o)}(-U)$. Since every dilation is an homeomorphism and b) holds, we have that $x \in \cap_{U\in\mathcal{F}(o)}U = \{0\}$. Hence, $x = 0$ and so $\overline{\{o\}} = \{o\}$, i.e. $\{o\}$ is closed.

c)⇒ a) Assume that X is not Hausdorff. Then by the previous proposition (2.1) does not hold, i.e. there exists $x \neq o$ s.t. $x \in U$ for all $U \in \mathcal{F}(o)$. This means that $x \in \cap_{U\in\mathcal{F}(o)}U \subseteq \cap_{U\in\mathcal{F}(o)\text{closed}}U = \overline{\{o\}}$ By c), $\overline{\{o\}} = \{o\}$ and so $x = 0$ which is a contradiction. □

Example 2.2.5. *Every vector space with an infinite number of elements endowed with the cofinite topology is not a tvs.* It is clear that in such topological space all singletons are closed (i.e. it is T1). Therefore, if it was a t.v.s. then by the previous results it should be a Hausdorff space which is not true as shown in Example 1.1.40.

2.3 Quotient topological vector spaces

Quotient topology

Let X be a topological space and $\sim$ be any equivalence relation on X. Then the *quotient set* $X/\sim$ is defined to be the set of all equivalence classes w.r.t. to $\sim$. The map $\phi : X \to X/\sim$ which assigns to each $x \in X$ its equivalence class $\phi(x)$ w.r.t. $\sim$ is called the *canonical map* or *quotient map*. Note that ϕ is surjective. We may define a topology on $X/\sim$ by setting that: a subset U of $X/\sim$ is open iff the preimage $\phi^{-1}(U)$ is open in X. This is called the *quotient topology* on $X/\sim$. Then it is easy to verify (Sheet 4, Exercise 2) that:

- the quotient map ϕ is continuous.
- the quotient topology on $X/\sim$ is the finest topology on $X/\sim$ s.t. ϕ is continuous.

Note that the quotient map ϕ is not necessarily open or closed.

Example 2.3.1. *Consider* $\mathbb{R}$ *with the standard topology given by the modulus and define the following equivalence relation on* $\mathbb{R}$*:*

$$x \sim y \Leftrightarrow (x = y \vee \{x, y\} \subset \mathbb{Z}).$$

Let $\mathbb{R}/\sim$ *be the quotient set w.r.t* $\sim$ *and* $\phi : \mathbb{R} \to \mathbb{R}/\sim$ *the correspondent quotient map. Let us consider the quotient topology on* $\mathbb{R}/\sim$*. Then* ϕ *is not an open map. In fact, if* U *is an open proper subset of* $\mathbb{R}$ *containing an integer, then* $\phi^{-1}(\phi(U)) = U \cup \mathbb{Z}$ *which is not open in* $\mathbb{R}$ *with the standard topology. Hence,* $\phi(U)$ *is not open in* $\mathbb{R}/\sim$ *with the quotient topology.*

For an example of quotient map which is not closed see Example 2.3.3 in the following.

Quotient vector space

Let X be a vector space and M a linear subspace of X. For two arbitrary elements $x, y \in X$, we define $x \sim_M y$ iff $x - y \in M$. It is easy to see that $\sim_M$ is an equivalence relation: it is reflexive, since $x - x = 0 \in M$ (every linear subspace contains the origin); it is symmetric, since $x - y \in M$ implies $-(x - y) = y - x \in M$ (if a linear subspace contains an element, it contains its inverse); it is transitive, since $x - y \in M$, $y - z \in M$ implies $x - z = (x - y) + (y - z) \in M$ (when a linear subspace contains two vectors, it also contains their sum). Then X/M is defined to be the quotient set $X/\sim_M$, i.e. the set of all equivalence classes for the relation $\sim_M$ described above. The canonical (or quotient) map $\phi : X \to X/M$ which assigns to each $x \in X$ its equivalence class $\phi(x)$ w.r.t. the relation $\sim_M$ is clearly surjective. Using the fact that M is a linear subspace of X, it is easy to check that:

1. if $x \sim_M y$, then $\forall \lambda \in \mathbb{K}$ we have $\lambda x \sim_M \lambda y$.
2. if $x \sim_M y$, then $\forall z \in X$ we have $x + z \sim_M y + z$.

These two properties guarantee that the following operations are well-defined on X/M:

- vector addition: $\forall \phi(x), \phi(y) \in X/M$, $\phi(x) + \phi(y) := \phi(x + y)$
- scalar multiplication: $\forall \lambda \in \mathbb{K}$, $\forall \phi(x) \in X/M$, $\lambda\phi(x) := \phi(\lambda x)$

X/M with the two operations defined above is a vector space and therefore it is often called *quotient vector space*. Then the quotient map ϕ is clearly linear.

Quotient topological vector space

Let X be now a t.v.s. and M a linear subspace of X. Consider the quotient vector space X/M and the quotient map $\phi : X \to X/M$ defined in Section 2.3. Since X is a t.v.s, it is in particular a topological space, so we can consider on X/M the quotient topology defined in Section 2.3. We already know that in this topological setting ϕ is continuous but actually the structure of t.v.s. on X guarantees also that it is open.

Proposition 2.3.2. *For a linear subspace M of a t.v.s. X, the quotient mapping $\phi : X \to X/M$ is open (i.e. carries open sets in X to open sets in X/M) when X/M is endowed with the quotient topology.*

Proof. Let V open in X. Then we have

$$\phi^{-1}(\phi(V)) = V + M = \cup_{m\in M}(V + m)$$

Since X is a t.v.s, its topology is translation invariant and so $V + m$ is open for any $m \in M$. Hence, $\phi^{-1}(\phi(V))$ is open in X as union of open sets. By definition, this means that $\phi(V)$ is open in X/M endowed with the quotient topology. □

It is then clear that ϕ carries neighborhoods of a point in X into neighborhoods of a point in X/M and viceversa. Hence, the neighborhoods of the origin in X/M are direct images under ϕ of the neighborhoods of the origin in X. In conclusion, when X is a t.v.s and M is a subspace of X, we can rewrite the definition of quotient topology on X/M in terms of neighborhoods as follows: *the filter of neighborhoods of the origin of X/M is exactly the image under ϕ of the filter of neighborhoods of the origin in X.*

It is not true, in general (not even when X is a t.v.s. and M is a subspace of X), that the quotient map is closed.

Example 2.3.3.
Consider $\mathbb{R}^2$ with the euclidean topology and the hyperbola $H := \{(x, y) \in \mathbb{R}^2 : xy = 1\}$. If M is one of the coordinate axes, then $\mathbb{R}^2/M$ can be identified with the other coordinate axis and the quotient map ϕ with the orthogonal projection on it. All these identifications are also valid for the topologies. The hyperbola H is closed in $\mathbb{R}^2$ but its image under ϕ is the complement of the origin on a straight line which is open.

Corollary 2.3.4. *For a linear subspace M of a t.v.s. X, the quotient space X/M endowed with the quotient topology is a t.v.s..*

Proof.
For convenience, we denote here by A the vector addition in X/M and just by $+$ the vector addition in X. Let W be a neighbourhood of the origin o in X/M. We aim to prove that $A^{-1}(W)$ is a neighbourhood of (o, o) in $X/M \times X/M$.

The continuity of the quotient map $\phi : X \to X/M$ implies that $\phi^{-1}(W)$ is a neighbourhood of the origin in X. Then, by Theorem 2.1.10-2 (we can apply the theorem because X is a t.v.s.), there exists V neighbourhood of the origin in X s.t. $V + V \subseteq \phi^{-1}(W)$. Hence, by the linearity of ϕ, we get $A(\phi(V) \times \phi(V)) = \phi(V + V) \subseteq W$, i.e. $\phi(V) \times \phi(V) \subseteq A^{-1}(W)$. Since ϕ is also open, $\phi(V)$ is a neighbourhood of the origin o in X/M and so $A^{-1}(W)$ is a neighbourhood of (o, o) in $X/M \times X/M$.

A similar argument gives the continuity of the scalar multiplication. □

Proposition 2.3.5. *Let X be a t.v.s. and M a linear subspace of X. Consider X/M endowed with the quotient topology. Then the two following properties are equivalent:*
a) M is closed
b) X/M is Hausdorff

Proof.
In view of Corollary 2.2.4, (b) is equivalent to say that the complement of the origin in X/M is open w.r.t. the quotient topology. But the complement of the origin in X/M is exactly the image under ϕ of the complement of M in X. Since ϕ is an open continuous map, the image under ϕ of the complement of M in X is open in X/M iff the complement of M in X is open, i.e. (a) holds. □

Corollary 2.3.6. *If X is a t.v.s., then $X/\overline{\{o\}}$ endowed with the quotient topology is a Hausdorff t.v.s.. $X/\overline{\{o\}}$ is said to be the* Hausdorff t.v.s. associated *with the t.v.s. X. When a t.v.s. X is Hausdorff, X and $X/\overline{\{o\}}$ are topologically isomorphic.*

Proof.
Since X is a t.v.s. and $\{o\}$ is a linear subspace of X, $\overline{\{o\}}$ is a closed linear subspace of X. Then, by Corollary 2.3.4 and Proposition 2.3.5, $X/\overline{\{o\}}$ is a Hausdorff t.v.s.. If in addition X is Hausdorff, then Corollary 2.2.4 guarantees that $\overline{\{o\}} = \{o\}$ in X. Therefore, the quotient map $\phi : X \to X/\overline{\{o\}}$ is also injective because in this case $Ker(\phi) = \{o\}$. Hence, ϕ is a topological isomorphism (i.e. bijective, continuous, open, linear) between X and $X/\overline{\{o\}}$ which is indeed $X/\{o\}$. □

2.4 Continuous linear mappings between t.v.s.

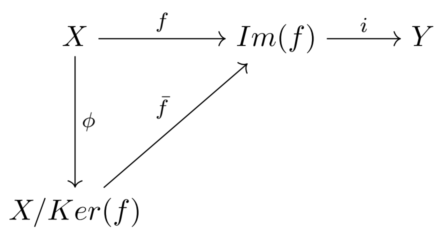
Let X and Y be two vector spaces over $\mathbb{K}$ and $f : X \to Y$ a linear map. We define the *image* of f, and denote it by $Im(f)$, as the subset of Y:

$$Im(f) := \{y \in Y \,:\, \exists\, x \in X \text{ s.t. } y = f(x)\}.$$

We define the *kernel* of f, and denote it by $Ker(f)$, as the subset of X:

$$Ker(f) := \{x \in X \,:\, f(x) = 0\}.$$

Both $Im(f)$ and $Ker(f)$ are linear subspaces of Y and X, respectively. We have then the diagram:



where i is the natural injection of $Im(f)$ into Y, i.e. the mapping which to each element y of $Im(f)$ assigns that same element y regarded as an element of Y; ϕ is the canonical map of X onto its quotient $X/Ker(f)$. The mapping $\bar{f}$ is defined so as to make the diagram commutative, which means that:

$$\forall\, x \in X,\, f(x) = \bar{f}(\phi(x)).$$

Note that

- $\bar{f}$ is well-defined.
  Indeed, if $\phi(x) = \phi(y)$, i.e. $x - y \in Ker(f)$, then $f(x - y) = 0$ that is $f(x) = f(y)$ and so $\bar{f}(\phi(x)) = \bar{f}(\phi(y))$.
- $\bar{f}$ is linear.
  This is an immediate consequence of the linearity of f and of the linear structure of $X/Ker(f)$.
- $\bar{f}$ is a one-to-one linear map of $X/Ker(f)$ onto $Im(f)$.
  The onto property is evident from the definition of $Im(f)$ and of $\bar{f}$. As for the one-to-one property, note that $\bar{f}(\phi(x)) = \bar{f}(\phi(y))$ means by definition that $f(x) = f(y)$, i.e. $f(x - y) = 0$. This is equivalent, by linearity of f, to say that $x-y \in Ker(f)$, which means that $\phi(x) = \phi(y)$.

The set of all linear maps (continuous or not) of a vector space X into another vector space Y is denoted by $\mathcal{L}(X;Y)$. Note that $\mathcal{L}(X;Y)$ is a vector space for the natural addition and multiplication by scalars of functions. Recall that when $Y=\mathbb{K}$, the space $\mathcal{L}(X;Y)$ is denoted by X^* and it is called the *algebraic dual* of X (see Definition 1.2.4).

Let us not turn to consider linear mapping between two t.v.s. X and Y. Since they posses a topological structure, it is natural to study in this setting continuous linear mappings.

Proposition 2.4.1. *Let $f: X \to Y$ a linear map between two t.v.s. X and Y. If Y is Hausdorff and f is continuous, then $Ker(f)$ is closed in X.*

Proof.
Clearly, $Ker(f) = f^{-1}(\{o\})$. Since Y is a Hausdorff t.v.s., $\{o\}$ is closed in Y and so, by the continuity of f, $Ker(f)$ is also closed in Y. □

Note that $Ker(f)$ might be closed in X also when Y is not Hausdorff. For instance, when $f \equiv 0$ or when f is injective and X is Hausdorff.

Proposition 2.4.2. *Let $f: X \to Y$ a linear map between two t.v.s. X and Y. The map f is continuous if and only if the map $\bar{f}$ is continuous.*

Proof.
Suppose f continuous and let U be an open subset in $Im(f)$. Then $f^{-1}(U)$ is open in X. By definition of $\bar{f}$, we have $\bar{f}^{-1}(U) = \phi(f^{-1}(U))$. Since the quotient map $\phi: X \to X/Ker(f)$ is open, $\phi(f^{-1}(U))$ is open in $X/Ker(f)$. Hence, $\bar{f}^{-1}(U)$ is open in $X/Ker(f)$ and so the map $\bar{f}$ is continuous. Viceversa, suppose that $\bar{f}$ is continuous. Since $f = \bar{f} \circ \phi$ and ϕ is continuous, f is also continuous as composition of continuous maps. □

In general, the inverse of $\bar{f}$, which is well defined on $Im(f)$ since $\bar{f}$ is injective, is not continuous. In other words, $\bar{f}$ is not necessarily bi-continuous.

The set of all continuous linear maps of a t.v.s. X into another t.v.s. Y is denoted by $L(X;Y)$ and it is a vector subspace of $\mathcal{L}(X;Y)$. When $Y=\mathbb{K}$, the space $L(X;Y)$ is usually denoted by X' which is called the *topological dual* of X, in order to underline the difference with X^* the algebraic dual of X. X' is a vector subspace of X^* and is exactly the vector space of all continuous linear functionals, or continuous linear forms, on X. The vector spaces X' and $L(X;Y)$ will play an important role in the forthcoming and will be equipped with various topologies.