A continuous moment problem for locally convex spaces.

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Outline

The classical moment problem in one dimension

Let μ be a non-negative Borel measure defined on $\mathbb{R}$. The $n-$**th moment of** μ is:

$$m_n^\mu := \int_{\mathbb{R}} x^n \mu(dx)$$

If all moments of μ exist and are finite, then $(m_n^\mu)_{n=0}^\infty$ is the **moment sequence** of μ.

μ non-neg. Borel measure	→ / ← ?	**Moment Sequence of μ**

Let $N \in \mathbb{N} \cup \{\infty\}$ and $K \subseteq \mathbb{R}$ closed.

The one-dimensional $K-$Moment Problem (MP)

Given a sequence $m = (m_n)_{n=0}^N$ of real numbers, does there exist a nonnegative Radon measure μ supported on a closed $K \subseteq \mathbb{R}$ s.t. for any $n = 0, 1, \ldots, N$ we have

$$m_n = \underbrace{\int_K x^n \mu(dx)}_{n\text{-th moment of } \mu} \quad ?$$

Remember: μ is **supported on** K if $\mu(\mathbb{R} \setminus K) = 0$.

$N = \infty \rightsquigarrow$ Full MP $\qquad N \in \mathbb{N} \rightsquigarrow$ Truncated MP

Riesz's Functional

Riesz's Functional

Let $m = (m_n)_{n=0}^{\infty}$ be such that $m_n \in \mathbb{R}$.

$$\begin{array}{rlcl} L_m : & \mathbb{R}[x] & \to & \mathbb{R} \\ & p(x) := \sum_{n=0}^{N} a_n \, x^n & \mapsto & L_m(p) := \sum_{n=0}^{N} a_n \, m_n. \end{array}$$

Note:

If m is represented by a non-negative measure μ on K, then

$$L_m(p) = \sum_{n=0}^{N} a_n \, m_n = \sum_{n=0}^{N} a_n \int_K x^n \mu(dx) = \int_K p(x)\mu(dx).$$

The one dimensional $K-$Moment Problem (MP)

Given a sequence $m = (m_n)_{n=0}^{\infty}$ of real numbers, does there exist a nonnegative Radon measure μ supported on a closed $K \subseteq \mathbb{R}$ s.t. for any $p \in \mathbb{R}[x]$ we have

$$L_m(p) = \int_K p(x)\mu(dx) \ ?$$

The classical $K-$moment problem in finite dimensions

Let $\mathbf{x} := (x_1, \dots, x_d)$ with $d \in \mathbb{N}$.

The d-dimensional $K-$Moment Problem (MP)

Given a linear functional $L : \mathbb{R}[\mathbf{x}] \to \mathbb{R}$, does there exist a nonnegative Radon measure μ supported on a closed $K \subseteq \mathbb{R}^d$ s.t. for any $p \in \mathbb{R}[\mathbf{x}]$ we have

$$L(p) = \int_K p(\mathbf{x})\mu(d\mathbf{x}) \ ?$$

- What if we have infinitely many variables?
- What if we take a generic $\mathbb{R}-$vector space V (even infinite dim.) instead of $\mathbb{R}^d$?
- What if we take a $\mathbb{R}-$algebra A instead of the polynomial ring $\mathbb{R}[\mathbf{x}]$?

$\rightsquigarrow$

Infinite dimensional K-Moment Problem (IMP)

Motivations for IMP: analysis of complex systems

Nervous system in which neural spikes travel

Fluid composed of molecules

Plants growing in a certain region

Dealing with MANY-BODY SYSTEMS

Galaxy composed of stars

STRATEGY:

- focus on few characteristics of the system
- predict the selected characteristics

OUTPUT: Putative characteristics of the system

INFINITE DIMENSIONAL MOMENT PROBLEM
Does there exist a real state of the original system which realizes the putative characteristics predicted?

A general formulation of MP

Terminology and Notations:

- $A = \mathbb{R}-$algebra, i.e. a $\mathbb{R}-$vector space with a bilinear product.
- $X(A) =$ character space of A, i.e. the set of all ring homomorphisms $\alpha : A \to \mathbb{R}$.
- For $a \in A$, $\hat{a} : X(A) \to \mathbb{R}$ is defined by $\hat{a}(\alpha) := \alpha(a)$ for all $\alpha \in X(A)$.
- $X(A)$ is given the weakest topology s.t. the functions $\hat{a}$, $a \in A$ are continuous.

The $K-$moment problem for $\mathbb{R}-$algebras

Given a linear functional $L : \mathbb{R}[\mathbf{x}] \to \mathbb{R}$, does there exist a nonnegative Radon measure μ supported on a Borel $K \subseteq X(\mathbb{R}[\mathbf{x}]) = \mathbb{R}^d$ s.t. for any $a \in \mathbb{R}[\mathbf{x}]$ we have

$$L(a) = \int_{X(\mathbb{R}[\mathbf{x}])} \hat{a}(\alpha)\mu(d\alpha) = \int_{\mathbb{R}^d} a(\alpha)\mu(d\alpha) \; ?$$

Remember that a measure μ is supported on a Borel $K \subseteq \mathbb{R}^d$ if $\mu(\mathbb{R}^d \setminus K) = 0$.

NB: Finite dimensional MP is a particular case

If $A = \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \ldots, x_d]$ then $X(A) = X(\mathbb{R}[\mathbf{x}])$ is identified (as tvs) with $\mathbb{R}^d$.
Ring homomorphisms $\mathbb{R}[\mathbf{x}] \to \mathbb{R}$ correspond to point evaluations $f \mapsto f(\alpha)$, $\alpha \in \mathbb{R}^d$ and so $X(\mathbb{R}[\mathbf{x}])$ corresponds to $\mathbb{R}^n$.

The $K-$moment problem for $\mathbb{R}-$algebras

The $K-$moment problem for $\mathbb{R}-$algebras

Given a linear functional $L : A \to \mathbb{R}$, does there exist a nonnegative Radon measure μ supported on a Borel $K \subseteq X(A)$ s.t. for any $a \in A$ we have

$$L(a) = \int_{X(A)} \hat{a}(\alpha)\mu(d\alpha)?$$

- **Pos(K)**$:= \{a \in A : \hat{a} \geq 0 \text{ on } K\}$
- M :=**2d-power module** generated by $p_1, \ldots, p_s \in A$
  $= \sum A^{2d} + p_1 \sum A^{2d} + \cdots + p_s \sum A^{2d}$ (M can be also infinitely generated!).
- $\mathbf{X_M} := \{\alpha \in X(A) : \hat{p}_i(\alpha) \geq 0,\ i = 1, \ldots, s\}$
- M **Archimedean** if $\forall\, a \in A$, $\exists\, N \in \mathbb{N}$: $N \pm a \in M$.

NOTE: If μ is a representing measure for L and $\text{supp}(\mu) \subseteq K$, then:

$$L(\text{Pos}(K)) \subseteq [0, +\infty) \text{ and in particular } L(M) \subseteq [0, +\infty).$$

What about the converse?

Thm (M. Ghasemi, M. Marshall, S. Wagner 2014; M. Ghasemi, S. Kuhlmann 2013)

Let M be an archimedean $2d-$power module of A and $L : A \to \mathbb{R}$ a linear functional.

$$\Big(L(M) \subseteq [0, +\infty)\Big) \Leftrightarrow \Big(\exists\, !\, \mu\text{: } \forall a \in A,\ L(a) = \int_{X(A)} \hat{a}(\alpha)\mu(d\alpha)\ \&\ \text{supp}(\mu) \subseteq X_M\Big).$$

Previous results for the infinite dimensional moment problem

IMP for linear functionals on the symmetric algebra of a lc *nuclear space*.

- Y. M. Berezansky, S. N. Sifrin, *Ukrain. Mat. Z.*, 1971.
- Y. M. Berezansky, Y. G. Kondratiev, *Naukova Dumka, Kiev*, 1988.
- H.J. Borchers, J. Yngvason, *Comm. Math. Phys.*, 1975.
- G.C. Hegerfeldt, *Comm. Math. Phys.*, 1975.
- T. Kuna, J. Lebowitz, E. Speer, *J.Stat. Phys.*, 2007– *Ann. Appl. Prob.*, 2011.
- M. Infusino, T. Kuna, A. Rota, *J. Funct. Analysis*, 2014.

IMP for continuous linear functionals on the symmetric algebra of a lc space.

- M. Ghasemi, M. Infusino, S. Kuhlmann, M. Marshall, Moment problem for symmetric algebras of locally convex spaces, *arXiv:1507.06781*, 2015.

IMP for continuous linear functionals on topological $\mathbb{R}-$algebras

- M. Ghasemi, S. Kuhlmann, *J. Funct. An.*, 2012.
- M. Ghasemi, S. Kuhlmann, M. Marshall, *J. Funct. An.*, 2014.
- M. Ghasemi, M. Marshall, S. Wagner *Can. Math. Bull.*, 2014.

IMP for linear functionals on $\mathbb{R}[x_i | i \in \Omega]$ when Ω is an arbitrary infinite index set.

- M. Ghasemi, S. Kuhlmann, M. Marshall, Moment problem in infinitely many variables, to appear in *Israel J. Math.*

Framework

- $V = \mathbb{R}$-vector space
- τ:= a **locally convex (lc)** topology on V
  = a topology on V generated by some family $\mathcal{S}$ of seminorms on V
  = the weakest topology on V s.t. each $\rho \in \mathcal{S}$ is continuous.
- $\rho : V \to [0, \infty)$ is a **seminorm** on V if
  (1) $\forall\ a \in V$ and $\forall\ r \in \mathbb{R}$, $\rho(ra) = |r|\rho(a)$,
  (2) $\forall\ a, b \in V$, $\rho(a + b) \leq \rho(a) + \rho(b)$.
- W.l.o.g. we assume that the family $\mathcal{S}$ is **directed**, i.e.
  $\forall\ \rho_1, \rho_2 \in \mathcal{S},\ \exists\ \rho \in \mathcal{S}\ \exists\, C > 0$ s.t. $C\rho(v) \geq \max\{\rho_1(v), \rho_2(v)\},\ \forall\ v \in V$.
- V^*:=**algebraic dual** of V=$\{\ell : V \to \mathbb{R} | \ell$ is a linear functional$\}$
- V':=**topological dual** of V=$\{\ell : V \to \mathbb{R} | \ell$ is a $\tau-$continuous linear functional$\}$
- $S(V)$= the **symmetric algebra** of V
  = the tensor algebra $T(V)$ factored by the ideal gen. by $v \otimes w - w \otimes v$
- $S(V)_k$=the k-**th homogeneous part** of $S(V)$
  = the image of k-th homogeneous part $V^{\otimes k}$ of $T(V)$ under the canonical map $\sum_{i=1}^n v_{i1} \otimes \cdots \otimes v_{ik} \mapsto \sum_{i=1}^n v_{i1} \cdots v_{ik}$.

IMP for symmetric algebras on a lc space

(V, τ) with τ lc-topology. Then:

- $X(S(V)) = Hom(S(V), \mathbb{R}) \cong V^* = Hom(V, R)$ via the isomorphism $\ell \mapsto \ell|_V$
- $\forall f \in S(V)$, $\hat{f} : X(S(V)) \to \mathbb{R}$ is given by $\alpha \mapsto \hat{f}(\alpha) := \alpha(f)$

The IMP for symmetric algebras on a lc space

Given a linear functional $L : S(V) \to \mathbb{R}$, does there exist a nonnegative Radon measure μ on V^* s.t. for any $f \in S(V)$ we have

$$L(f) = \int_{V^*} \hat{f}(\alpha)\mu(d\alpha)\,?$$

Is μ unique? What is the support of μ?

QUESTION: continuous functionals

What happens when $L : S(V) \to \mathbb{R}$ is continuous?
Which topology is natural to consider on $S(V)$? Can τ on V be extended to $S(V)$?

Continuous functionals on $S(V)$ (I. case)

(I. case): τ is generated by $S = \{\rho\}$, i.e. (V, ρ) with ρ seminorm on V.

Proposition (Ghasemi, I., Kuhlmann, Marhsall, 2015)

Any seminorm ρ on V can be extended to a **submultiplicative seminorm** $\overline{\rho}$ on $S(V)$, i.e. $\overline{\rho}(fg) \leq \overline{\rho}(f)\overline{\rho}(g)$, $\forall f, g \in S(V)$.

1. **tensor seminorm $\rho^{\otimes k}$ on $V^{\otimes k}$**:
$$(\rho^{\otimes k})(f) := \inf\{\sum_{i=1}^{n} \rho(f_{i1}) \cdots \rho(f_{ik}) : f = \sum_{i=1}^{n} f_{i1} \otimes \cdots \otimes f_{ik},\ f_{ij} \in V,\ n \geq 1\}.$$
2. Let $\pi_k : V^{\otimes k} \to S(V)_k$ be the canonical map.
For $k \geq 1$ define $\overline{\rho}_k$ to be the **quotient seminorm on $S(V)_k$** induced by $\rho^{\otimes k}$:
$$\overline{\rho}_k(f) = \inf\{\sum_{i=1}^{n} \rho(f_{i1}) \cdots \rho(f_{ik}) : f = \sum_{i=1}^{n} f_{i1} \cdots f_{ik}, f_{ij} \in V, n \geq 1\}.$$
Define $\overline{\rho}_0$ to be the usual absolute value on $\mathbb{R}$.
3. Extend ρ to a submultiplicative seminorm $\overline{\rho}$ on $S(V)$ by taking the **projective extension of ρ to $S(V)$** defined for any $f = f_0 + \cdots + f_r$, $f_k \in S(V)_k$, $k = 0, \ldots, r$ by:
$$\overline{\rho}(f) := \sum_{k=0}^{r} \overline{\rho}_k(f_k).$$

Continuous functionals on $S(V)$ (I. case)

Proposition

(V, ρ) s.t. ρ seminorm

$\downarrow$

$(S(V), \overline{\rho})$ s.t. $\overline{\rho}$ submult. seminorm

Thm (Ghasemi, Kuhlmann, Marshall, 2014)

Let (A, σ) be a submult. seminormed $\mathbb{R}$-alg. and M a $2d$-power module of A. If $L : A \to \mathbb{R}$ is a $\sigma-$continuous linear functional, then:

$$\Big(L(M) \subseteq [0, +\infty)\Big) \Leftrightarrow \Big(\exists\,!\,\mu \text{ on } X(A): L(a) = \int_{X(A)} \hat{a}(\alpha)\mu(d\alpha) \;\&\; \operatorname{supp}\mu \subseteq X_M \cap \mathfrak{sp}(\sigma)\Big)$$

$\searrow$ $\swarrow$

Theorem (Ghasemi, I., Kuhlmann, Marhsall, 2015)

Let (V, ρ) be a seminormed $\mathbb{R}$-vector space and M be a $2d$-power module of $S(V)$. If $L : S(V) \to \mathbb{R}$ is a $\overline{\rho}$-continuous linear functional, then:

$$\Big(L(M) \subseteq [0, +\infty)\Big) \Leftrightarrow \Big(\exists\,!\,\mu \text{ on } V^*:\; L(f) = \int_{V^*} \hat{f}(\alpha)\mu(d\alpha) \;\&\; \operatorname{supp}\mu \subseteq X_M \cap \overline{B}_1(\rho')\Big)$$

Notation:

-**Gelfand spectrum** of ρ: $\mathfrak{sp}(\rho) := \{\alpha \in X(A) : \alpha \text{ is } \rho\text{-continuous}\}$.

-**operator norm in** V^* w.r.t. ρ: $\rho'(v^*) := \inf\{C \in [0, \infty) : |v^*(f)| \leq C\rho(f)\ \forall f \in V\}$

-**closed unitary ball** in V^* w.r.t. ρ: $\overline{B}_1(\rho') := \{v^* \in V^* : \rho'(v^*) \leq 1\}$.

Continuous functionals on $S(V)$ (II. case)

(II. case): τ is a lc topology on V generated by a directed family $\mathcal{S}$ of seminorms

Lemma

Suppose that τ is a lc topology on V generated by a directed family $\mathcal{S}$ of seminorms.
$\Big(L : V \to \mathbb{R}$ is τ-continuous $\Big) \Leftrightarrow \Big(\exists\, \rho \in \mathcal{S}$ s.t. L is ρ-continuous$\Big)$.

Proposition (Ghasemi, I., Kuhlmann, Marhsall, 2015)

(V, τ) s.t. τ lc topology $\to$ $(S(V), \overline{\tau})$ s.t. $\overline{\tau}$ lmc topology

-$\overline{\tau}$ defined by the directed family of submultiplicative seminorms $\overline{i\rho}$, $\rho \in \mathcal{S}$, $i \in \mathbb{N}$)
-$\overline{\tau}$ is the finest lmc topology on $S(V)$.

RECALL: A **locally multiplicatively convex** (lmc) topology on an $\mathbb{R}$-algebra A is a topology on A generated by some family of submultiplicative seminorms on A.

Theorem (Ghasemi, I., Kuhlmann, Marhsall, 2015)

Let (V, τ) be a lc $\mathbb{R}$-vector space and M be a $2d$-power module of $S(V)$.
If $L : S(V) \to \mathbb{R}$ is a $\overline{\tau}$-continuous linear functional, then:

$$\Big(L(M) \subseteq [0, +\infty)\Big) \Leftrightarrow \Big(\exists!\, \mu \text{ on } V^*\text{: } L(f) = \int_{V^*} \hat{f}(\alpha)\mu(d\alpha) \;\&\; \operatorname{supp}\mu \subseteq X_M \cap \overline{B}_i(\rho') \text{ for some } \rho \in \mathcal{S} \text{ and some integer } i \geq 1\Big)$$

Previous results on lc nuclear spaces

(V, τ) is assumed to be:

- **separable**
- **projective limit** of a directed family $(H_s)_{s \in S}$ of **Hilbert spaces**
- **nuclear**: $\forall s_1 \in S \; \exists s_2 \in S$ s.t. $H_{s_2} \hookrightarrow H_{s_1}$ is quasi-nuclear.
- $\tau=$ **projective topology**: lc topology on V defined by the directed family $\mathcal{S}$ of norms of V which are induced by the embeddings $V \hookrightarrow H_s$, $s \in S$.

Theorem (Berezansky, Kondratiev 1988–Berezansky, Sifrin 1975)

Let (V, τ) be a nuclear space as above and $L : S(V) \to \mathbb{R}$ a linear functional. If

1. $L(\sum S(V)^2) \subseteq [0, \infty)$
2. for each $k \geq 0$ the restriction map $L : S(V)_k \to \mathbb{R}$ is continuous w.r.t. the lc topology $\overline{\tau}_k$ on $S(V)_k$ induced by the norms $\{\overline{\rho}_k : \rho \in \mathcal{S}\}$;
3. $\exists\, E \subset V$ countable with span(E) dense in (V, τ) s.t. $C\{m_k\}$ is quasi-analytic, where:

$$m_0 := \sqrt{L(1)}, \text{ and } m_k := \sqrt{\sup_{f_1,\dots,f_{2k} \in E} |L(f_1 \dots f_{2k})|}, \text{ for } k \geq 1$$

Then $\exists!$ μ on V^* s.t. $\forall\, f \in S(V)$, $L(f) = \int \hat{f}(\alpha)\mu(d\alpha)$ & supp$(\mu) \subseteq V'$.

Notation: (2)+(3)= L is **determining** In **[I., Kuna, Rota, 2014]** an improvement of this theorem is given for $V = \mathcal{C}_c^\infty(\mathbb{R}^d)$.

Comparison with previous results on lc nuclear spaces

Ghasemi, I., Kuhlmann, Marhsall	Berezansky, Kondratiev, Sifrin	I., Kuna, Rota
(V,τ) lc	(V,τ) lc and nuclear	$(\mathcal{C}_c^\infty(\mathbb{R}^d), \tau_{proj})$ lc and nuclear
$S(V)$	$\mathcal{P}_V(V') \cong S(V)$	$\mathcal{P}_{\mathcal{C}_c^\infty}(\mathscr{D}'_{proj}) \cong S(\mathcal{C}_c^\infty(\mathbb{R}^d))$
L is $\overline{\tau}$-continuous on $S(V)$	L is determining	L is determining
M=2d-power module of $S(V)$	$M = \sum S(V)^2$	Q=quadratic module of $S(V)$
$\mathrm{supp}(\mu) \subseteq X_M \cap \overline{B}_i(\rho') \subseteq X_M \cap V'$	$\mathrm{supp}(\mu) \subseteq V'$	$\mathrm{supp}(\mu) \subseteq X_Q \cap V'$

Open questions and work in progress

- Can we generalize our main result by assuming only the determining condition on L?
- Can we generalize our main result by assuming only the continuity of L on each $S(V)_k$?
- Would this still give a better characterization of the support than the previous results on the nuclear spaces?

For more details see:

M. Ghasemi, M. Infusino, S. Kuhlmann, M. Marshall, **Moment problem for symmetric algebras of locally convex spaces**, arXiv:1507.06781.

Thank you for your attention

and

Thank you Murray

working with you was an incomparable opportunity for me.
I miss you.

Newton Institute, Cambridge-July, 2013.