Rigorous Navier-Stokes Limit of the Lattice Boltzmann Equation

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Abstract

In this article, we rigorously investigate the diffusive limit of a velocity-discrete Boltzmann equation which is used in the lattice Boltzmann method (LBM) to construct approximate solutions of the incompressible Navier-Stokes equation. Our results apply to LBM collision operators with multiple collision frequencies (generalized lattice Boltzmann) which include the widely used BGK operators.

Keywords. lattice Boltzmann method, Navier-Stokes equation, multiple collision frequencies, diffusive scaling, stability, incompressible limit
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1 Introduction

In this article, we are concerned with a velocity-discrete Boltzmann equation in the diffusive scaling

$$\frac{\partial f_i}{\partial t} + \frac{1}{\epsilon}\boldsymbol{c}_i \cdot \nabla f_i = \frac{1}{\epsilon^2} J_i(f), \qquad i = 0, \dots, N, \tag{1}$$

which arises in connection with a numerical method for the incompressible Navier-Stokes equation, the so-called lattice Boltzmann method (LBM) [6, 8]. The system (1) describes the evolution of a hypothetical gas or liquid in which the atoms can only travel with velocities from the discrete set $\mathcal{V} = \{\boldsymbol{c}_0, \dots, \boldsymbol{c}_N\}$. The particle densities f_i specify how many particles have the velocity $\boldsymbol{c}_i \in \mathcal{V}$ at time $t \geq 0$ and position $\boldsymbol{x} \in \Omega$. While the left-hand side in (1) describes the transport of the particles, the right-hand side models interaction of the particles by collisions.

Before we specify details of the structure of $\mathcal{V}$ and J, let us briefly mention how the lattice Boltzmann method is related to (1) (for more details, see [14, 15]). Integrating (1) along characteristics, we find

$$f_i(t + \Delta t, \boldsymbol{x} + \boldsymbol{c}_i \Delta t/\epsilon) = f_i(t, \boldsymbol{x}) + \int_0^{\Delta t} \frac{1}{\epsilon^2} J_i(f)(t + s, \boldsymbol{x} + \boldsymbol{c}_i s/\epsilon)\, ds.$$

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Setting $\epsilon = \Delta x, \Delta t = \epsilon \Delta x$, and approximating the integral by the simple rectangle rule with evaluation at the left point of the interval, we obtain

$$f_i(t + \Delta t, \boldsymbol{x} + \boldsymbol{c}_i \Delta x) = f_i(t, \boldsymbol{x}) + J_i(f)(t, \boldsymbol{x}), \tag{2}$$

which is exactly the lattice Boltzmann evolution. If the discrete velocity set $\mathcal{V}$ is chosen in such a way that the set of all integer linear combinations forms a regular lattice $\mathcal{X} = \{\sum_i n_i \boldsymbol{c}_i : n_i \in \mathbb{Z}\}$, then (1) is already completely discretized if $\boldsymbol{x}$ is restricted to $\Delta x\, \mathcal{X}$. Under suitable conditions on the initial values for (1), it turns out that the average $\boldsymbol{u} = \sum_i \boldsymbol{c}_i f_i / \epsilon$ is an approximate solution of the incompressible Navier-Stokes equation

$$\operatorname{div} \boldsymbol{u} = 0, \qquad \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \nu \Delta \boldsymbol{u}, \qquad \boldsymbol{u}|_{t=0} = \bar{\boldsymbol{u}}. \tag{3}$$

This relation is usually justified by carrying out a formal Chapman-Enskog expansion in ϵ (see, e.g., [11]).
In this article, our aim is to give a *rigorous* justification of the relation between the continuous version (1) of the lattice Boltzmann equation and the Navier-Stokes equation (3). This is a classical subject on the diffusive limit of discrete velocity kinetic equations [1, 3, 4, 5]. Our analysis will be in the spirit of [9, 1, 2, 10, 3, 4, 5] but it differs from these results because we concentrate on the collision operators which are used in the lattice Boltzmann method. Our problem is also different from those in [23, 19, 25] because our limit system consists of incompressible Navier-Stokes equations.
To fix ideas, we will work in a specific two-dimensional situation but the ideas can be transfered to other models and three dimensions. The spatial domain Ω will be the unit torus (i.e. the unit square with periodic boundary conditions) and the velocity set is chosen as in the D2Q9 model (nine velocities in two space dimensions – see Fig. 1) where $\mathcal{V} = \{\boldsymbol{c}_0, \ldots, \boldsymbol{c}_8\}$ with $\boldsymbol{c}_0 = \boldsymbol{0}$ and

$$\boldsymbol{c}_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \qquad \boldsymbol{c}_2 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \qquad \boldsymbol{c}_3 = \left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right) \qquad \boldsymbol{c}_4 = \left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right)$$
$$\boldsymbol{c}_5 = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \qquad \boldsymbol{c}_6 = \left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right) \qquad \boldsymbol{c}_7 = \left(\begin{smallmatrix}-1\\-1\end{smallmatrix}\right) \qquad \boldsymbol{c}_8 = \left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)$$

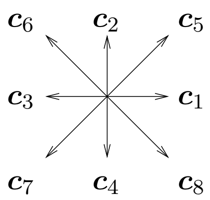

Figure 1: Discrete velocities in the D2Q9 model

To simplify notation, we introduce the Euclidean vector space $\mathcal{F}$ of real valued functions $f : \mathcal{V} \to \mathbb{R}$ with the canonical scalar product

$$\langle f, g \rangle = \sum_{\boldsymbol{v} \in \mathcal{V}} f(\boldsymbol{v}) g(\boldsymbol{v}), \qquad f, g \in \mathcal{F}.$$

With the multiplication operators $V_1, V_2 : \mathcal{F} \to \mathcal{F}$ defined by $(V_i f)(\boldsymbol{v}) = v_i f(\boldsymbol{v})$, we can form the vector $\boldsymbol{V} = (V_1, V_2)^T$ and rewrite (1) as equation for $f(t, \boldsymbol{x}, \boldsymbol{c}_i) = f_i(t, \boldsymbol{x})$:

$$\frac{\partial f}{\partial t} + \frac{1}{\epsilon}\boldsymbol{V} \cdot \nabla f = \frac{1}{\epsilon^2} J(f). \tag{4}$$

The collision operator $J : \mathcal{F} \to \mathcal{F}$ is chosen as

$$J(f) = \mathcal{A}(f^{eq}(f) - f)$$

where $\mathcal{A} : \mathcal{F} \to \mathcal{F}$ is a linear mapping (with properties specified in section 2) and $f^{eq} : \mathcal{F} \to \mathcal{F}$ is the so-called equilibrium distribution which we choose as in [11]. It depends linearly on the average density $\rho = \langle 1, f \rangle$ and quadratically on the average momentum $\bar{\rho}\boldsymbol{u} = \langle 1, \boldsymbol{V} f \rangle$

$$f^{eq}(f) = F^{eq}_{\bar{\rho}}(\langle 1, f \rangle, \langle 1, \boldsymbol{V} f \rangle)$$

with

$$F^{eq}_{\bar{\rho}}(\rho, \boldsymbol{u}; \boldsymbol{v}) = \big(\rho + 3\bar{\rho}\boldsymbol{u} \cdot \boldsymbol{v} + \bar{\rho}(3\boldsymbol{u} \cdot \boldsymbol{v})^2/2 - 3\bar{\rho}|\boldsymbol{u}|^2/2\big) f^*(\boldsymbol{v}).$$

The function $f^* = F^{eq}_{\bar{\rho}}(1, \boldsymbol{0})$ is defined by

$$f^*(\mathbf{c}_i) = \begin{cases} 4/9, & i = 0 \\ 1/9, & i = 1, 2, 3, 4 \\ 1/36, & i = 5, 6, 7, 8. \end{cases}$$

In the following, we set $\bar{\rho} = 1$. As initial data, we choose a small perturbation of the function f^*

$$f|_{t=0} = F^{eq}_1(1, \epsilon\bar{\boldsymbol{u}}) + O(\epsilon^2) \tag{5}$$

where $\bar{\boldsymbol{u}} : \Omega \to \mathbb{R}^2$ is a smooth, divergence-free velocity field. The $O(\epsilon^2)$-term will be specified later to avoid initial-layers. We remark that in lattice Boltzmann simulations, the initial value typically lacks this correction, i.e. $f|_{t=0} = F^{eq}_1(1, \epsilon\bar{\boldsymbol{u}})$ but we postpone a careful investigation of the resulting initial-layer effects to some future work.

With a careful analysis involving Hilbert expansions, we prove that the solution f^ϵ of the initial value problem (4), (5) has the property that $\boldsymbol{u}_\epsilon/\epsilon = \langle 1, f^\epsilon \rangle / \epsilon$ differs only by terms of order ϵ^2 from the solution $\boldsymbol{u}$ of the incompressible Navier-Stokes equation (3) with a viscosity parameter ν being related to an eigenvalue of the operator $\mathcal{A}$. This result is more precise (for our models) than those in [1, 3, 4, 5]. See Theorem 4.1 (Section 4) for details.

2 Collision operator and stability structure

The idea to use collision operators of relaxation type $J(f) = \mathcal{A}(f^{eq}(f) - f)$ in the lattice Boltzmann approach goes back to [12, 20]. In the multiple-relaxation-time (or generalized) lattice Boltzmann model [13], the operator $\mathcal{A}$ is set up by specifying an orthonormal basis of $\mathcal{F}$ and assuming that $\mathcal{A}$ diagonalizes in this basis. This approach has shown to be more flexible and stable [18] than the widely used BGK collision operator, where $\mathcal{A}$ is a multiple of the identity [21, 7]. In the following, we will adopt the multiple-relaxation-time model where the linear operator $\mathcal{A}$ is essentially determined by certain algebraic properties (which reflect physical symmetries and conservation): $\mathcal{A}$ should be symmetric and positive semidefinite, it should commute with 90° rotations and reflections, and the kernel should be generated by the functions $p_1(\boldsymbol{v}) = 1$, $p_2(\boldsymbol{v}) = v_1$, $p_3(\boldsymbol{v}) = v_2$. Introducing rotation and reflection by the operators $(Rf)(\boldsymbol{v}) = f(-v_2, v_1)$, $(Sf)(\boldsymbol{v}) = f(-v_1, v_2)$, these properties can be written as

i. $\langle \mathcal{A}f, g\rangle = \langle f, \mathcal{A}g\rangle$

ii. $\mathcal{A}R = R\mathcal{A}$, $\mathcal{A}S = S\mathcal{A}$

iii. $\mathcal{A}$ is positive semidefinite

iv. $\{1, v_1, v_2\}$ generates the kernel of $\mathcal{A}$

It is possible to completely characterize operators $\mathcal{A}$ which satisfy conditions (i) to (iv). This investigation is easily carried out in an orthonormal basis $\{q_1, \dots, q_9\}$ of $\mathcal{F}$ related to eigenvectors of the rotation operator R and the reflection operator S (up to the signs of q_6, q_7, q_8 this is the basis used in [13]):

$$q_1(\boldsymbol{v}) = \frac{1}{3},$$
$$q_2(\boldsymbol{v}) = \frac{v_1}{\sqrt{6}}, \qquad q_3(\boldsymbol{v}) = \frac{v_2}{\sqrt{6}},$$
$$q_4(\boldsymbol{v}) = \frac{v_1 v_2}{2}, \qquad q_5(\boldsymbol{v}) = \frac{v_1^2 - v_2^2}{2}, \qquad q_6(\boldsymbol{v}) = \frac{3v_1^2 + 3v_2^2 - 4}{6},$$
$$q_7(\boldsymbol{v}) = \frac{3v_1 v_2^2 - 2v_1}{2\sqrt{3}}, \qquad q_8(\boldsymbol{v}) = \frac{3v_1^2 v_2 - 2v_2}{2\sqrt{3}},$$
$$q_9(\boldsymbol{v}) = \frac{3}{2}v_1^2 v_2^2 - v_1^2 - v_2^2 + \frac{2}{3}.$$

Introducing the orthogonal projectors $(Q_i f) = \langle f, q_i\rangle\, q_i$, we can write (the majority of) all operators satisfying (i) to (iv) in the form

$$\mathcal{A} = \sum_{i=1}^{9} \lambda_i Q_i, \qquad \lambda_1, \lambda_2, \lambda_3 = 0, \quad \lambda_7 = \lambda_8, \quad \lambda_4, \dots, \lambda_9 > 0. \tag{6}$$

For later use, we introduce the projection $Q = \sum_{i=1}^{3} Q_i$ onto the kernel of $\mathcal{A}$. Note that

$$Qf = 0 \qquad \Longleftrightarrow \qquad \langle 1, f\rangle = 0, \quad \langle 1, \boldsymbol{V} f\rangle = \boldsymbol{0}. \tag{7}$$

Since the remaining eigenvalues are strictly positive, we can define a pseudo inverse of $\mathcal{A}$ using $P = I - Q$

$$\mathcal{A}^\dagger := \left(\mathcal{A}|_{P(\mathcal{F})}\right)^{-1} P.$$

Note that $\mathcal{A}^\dagger : \mathcal{F} \to \mathcal{F}$ satisfies

$$Q\mathcal{A}^\dagger = \mathcal{A}^\dagger Q = 0, \qquad P\mathcal{A}^\dagger = \mathcal{A}^\dagger P = \mathcal{A}^\dagger, \qquad \mathcal{A}\mathcal{A}^\dagger = P = \mathcal{A}^\dagger \mathcal{A}. \tag{8}$$

Another important property of $P, Q, \mathcal{A}$, and $\mathcal{A}^\dagger$ is the preservation of even/odd symmetry. The reason is that all these operators commute with double rotations R^2 – the building block for the odd/even projections

$$S_e = \frac{1}{2}(I + R^2), \qquad S_o = \frac{1}{2}(I - R^2)$$

(note that $(S_e f)(\boldsymbol{v}) = (f(\boldsymbol{v}) + f(-\boldsymbol{v}))/2$, and $(S_o f)(\boldsymbol{v}) = (f(\boldsymbol{v}) - f(-\boldsymbol{v}))/2$). For example, we have

$$S_e \mathcal{A} = (\mathcal{A} + R^2 \mathcal{A})/2 = \mathcal{A}(I + R^2)/2 = \mathcal{A} S_e$$

so that $\mathcal{A}$ applied to some even function f (i.e. $f = S_e f$) is again even: $S_e \mathcal{A} f = \mathcal{A} S_e f = \mathcal{A} f$. Similarly, one shows that $P, Q, \mathcal{A}$, and $\mathcal{A}^\dagger$ commute with S_e, S_o and thus preserve odd/even symmetry.
Next, we consider the equilibrium distribution $f^{eq}(f) = F_1^{eq}(\langle 1, f \rangle, \langle 1, \boldsymbol{V} f \rangle)$ which can be split into a linear and a quadratic part

$$f^{eq}(f) = f^{eq}_{lin}(f) + f^{eq}_{quad}(f, f) \tag{9}$$

where

$$f^{eq}_{lin}(f) = F^{eq}_{lin}(\langle 1, f \rangle, \langle 1, \boldsymbol{V} f \rangle), \qquad f^{eq}_{quad}(f, g) = F^{eq}_{quad}(\langle 1, \boldsymbol{V} f \rangle, \langle 1, \boldsymbol{V} g \rangle)$$

and

$$F^{eq}_{lin}(\rho, \boldsymbol{u}) = (\rho + 3\boldsymbol{u} \cdot \boldsymbol{V}) f^*,$$
$$F^{eq}_{quad}(\boldsymbol{u}, \boldsymbol{w}) = ((3\boldsymbol{u} \cdot \boldsymbol{V})(3\boldsymbol{w} \cdot \boldsymbol{V})/2 - 3\boldsymbol{u} \cdot \boldsymbol{w}/2) f^*.$$

The functions $F^{eq}_{lin}, F^{eq}_{quad}$ are constructed in such a way that

$$\left\langle 1, F^{eq}_{lin}(\rho, \boldsymbol{u}) \right\rangle = \rho, \quad \left\langle 1, \boldsymbol{V} F^{eq}_{lin}(\rho, \boldsymbol{u}) \right\rangle = \boldsymbol{u}, \quad \left\langle 1, V_i V_j F^{eq}_{lin}(\rho, \boldsymbol{u}) \right\rangle = \frac{\rho}{3} \delta_{ij} \tag{10}$$

and

$$\left\langle 1, (1, \boldsymbol{V}) F^{eq}_{quad}(\boldsymbol{u}, \boldsymbol{w}) \right\rangle = (0, \boldsymbol{0}), \quad \left\langle 1, V_i V_j F^{eq}_{quad}(\boldsymbol{u}, \boldsymbol{w}) \right\rangle = \frac{u_i w_j + u_j w_i}{2}. \tag{11}$$

Inspecting the definition of F^{eq}_{quad}, it easily follows that

$$f^{eq}_{quad}(f, g) = S_e f^{eq}_{quad}(S_o f, S_o g), \qquad f^{eq}_{quad}(f, 0) = 0. \tag{12}$$

Note that, in view of (7) and (10), we have $Q(f^{eq}(f) - f) = 0$ so that for any $\tau > 0$, $\mathcal{A}(f^{eq}(f) - f) = (Q/\tau + \mathcal{A})(f^{eq}(f) - f)$. Hence, if we choose $\lambda_4 = \cdots = \lambda_9 = 1/\tau$, we obtain the so-called BGK collision operator $J(f) = (f^{eq}(f) - f)/\tau$ which is frequently used in LBM [21, 7, 8] and which will be covered by our considerations.
Introducing the linear part of the collision operator

$$J_0 = \mathcal{A}(f^{eq}_{lin} - I), \tag{13}$$

we can write

$$J(f) = J_0 f + \mathcal{A} f^{eq}_{quad}(f, f). \tag{14}$$

We remark that the directional derivative of J at the point $f \in \mathcal{F}$ in direction $h \in \mathcal{F}$ is given by

$$DJ(f)h = J_0 h + 2\mathcal{A} f^{eq}_{quad}(f, h). \tag{15}$$

Finally, let us concentrate on the stability structure of equation (2). We observe that (4) is a symmetric hyperbolic system. However, for stability reasons, we require also some symmetry and definiteness properties of the right-hand side. To achieve this, we need the following result.

Lemma 2.1
Let B_0 be the positive definite multiplication operator $B_0 f = f/f^*$ and let $\mathcal{A}$ be of the form (6) with $\lambda_6 = \lambda_9$. Then there exist linear operators $P_k : \mathcal{F} \to \mathcal{F}$, $k = 1, \ldots, 9$ with adjoints P_k^* such that

$$P_i^* P_k = \delta_{ik} P_k^* P_k, \qquad B_0 = \sum_{k=1}^{9} P_k^* P_k, \qquad B_0 J_0 = -\sum_{k=1}^{9} \lambda_k P_k^* P_k. \tag{16}$$

and

$$P_i J_0 = 0, \quad P_i J(f) = 0, \qquad i = 1, 2, 3. \tag{17}$$

Proof: We first show that $B_0 J_0$ is a symmetric operator. To see this, we introduce the orthogonal subspaces

$U_4 = \text{span}\{q_4\}$, $U_5 = \text{span}\{q_5\}$, $U_6 = \text{span}\{q_1, q_6, q_9\}$, $U_7 = \text{span}\{q_2, q_3, q_7, q_8\}$,

and note that $3f^* = q_1 - q_6 + q_9/2 \in U_6$, $V_i f^* \in U_7$, and $B_0 U_k \subset U_k$. Moreover, because of the assumption $\lambda_6 = \lambda_9$, we have $\mathcal{A} U_k \subset U_k$. Using a subscript k on $f \in \mathcal{F}$ to denote the projection onto U_k, we find

$$f^{eq}(f) - f = -f_4 - f_5 - (f_6 - \langle 1, f_6 \rangle f_6^*) - (f_7 - 3\langle 1, \boldsymbol{V} f_7 \rangle \cdot (\boldsymbol{V} f^*)_7).$$

Since

$$\langle f_6 - \langle 1, f_6 \rangle f_6^*, q_1 \rangle = 0, \qquad \langle f_7 - 3\langle 1, \boldsymbol{V} f_7 \rangle \cdot (\boldsymbol{V} f^*)_7, q_i \rangle = 0, \quad i = 2, 3$$

we conclude

$$J_0 f = -\lambda_4 f_4 - \lambda_5 f_5 - \lambda_6 (f_6 - \langle 1, f_6 \rangle f^*) - \lambda_7 (f_7 - 3\langle 1, \boldsymbol{V} f_7 \rangle \cdot \boldsymbol{V} f^*). \tag{18}$$

To show symmetry of $B_0 J_0$ we recall that U_k are invariant, orthogonal subspaces of $B_0 J_0$ so that

$$\langle B_0 J_0 f, g \rangle = \sum_{k=4}^{7} \langle B_0 J_0 f_k, g_k \rangle.$$

Thus, it remains to show that $\langle B_0 J_0 f_k, g_k \rangle$ are symmetric expressions in g_k and f_k. For $k = 4, 5$ this follows immediately from

$$\langle B_0 J_0 f_k, g_k \rangle = -\lambda_k \langle B_0 f_k, g_k \rangle = -\lambda_k \left\langle B_0^{\frac{1}{2}} f_k, B_0^{\frac{1}{2}} g_k \right\rangle, \quad k = 4, 5.$$

For the other subspaces, we have with $B_0 f^* = 1$

$$\langle B_0 J_0 f_6, g_6 \rangle = -\lambda_6 \left\langle B_0^{\frac{1}{2}} f_6, B_0^{\frac{1}{2}} g_6 \right\rangle + \lambda_6 \langle 1, f_6 \rangle \langle 1, g_6 \rangle ,$$

and

$$\langle B_0 J_0 f_7, g_7 \rangle = -\lambda_7 \left\langle B_0^{\frac{1}{2}} f_7, B_0^{\frac{1}{2}} g_7 \right\rangle + \lambda_7 \langle 1, \boldsymbol{V} f_7 \rangle \cdot \langle 1, \boldsymbol{V} g_7 \rangle .$$

We remark that the eigenvectors and eigenvalues of J_0 can easily be read off from (18). Obvious eigenvectors are $q_4, \ldots, q_9$ with eigenvalues $-\lambda_4, \ldots, -\lambda_9$. The remaining three eigenvectors belong to the eigenvalue zero: f^*, $V_1 f^*$, $V_2 f^*$. Using the fact that

$$B_0^{\frac{1}{2}} J_0 B_0^{-\frac{1}{2}} = B_0^{-\frac{1}{2}} B_0 J_0 B_0^{-\frac{1}{2}}$$

is symmetric and has the same eigenvalues $-\lambda_k$ as the operator J_0, we can find a basis of orthonormal eigenvectors r_k. Using the corresponding orthogonal projectors $R_k f = \langle f, r_k \rangle r_k$, we have $\sum_{k=1}^{9} R_k = I$, and

$$B_0^{\frac{1}{2}} J_0 B_0^{-\frac{1}{2}} = -\sum_{k=1}^{9} \lambda_k R_k.$$

Defining $P_k = R_k B_0^{\frac{1}{2}}$, relations (16) follow immediately. Finally, to show (17), we observe for $i = 1, 2, 3$

$$P_i J_0 = R_i B_0^{\frac{1}{2}} J_0 B_0^{-\frac{1}{2}} B_0^{\frac{1}{2}} = -\sum_{k=1}^{9} \lambda_k R_i R_k B_0^{\frac{1}{2}} = -\lambda_i R_i B_0^{\frac{1}{2}} = 0.$$

In view of (14)

$$P_i J(f) = P_i J_0 f + P_i \mathcal{A} f^{eq}_{quad}(f,f) = P_i \mathcal{A} f^{eq}_{quad}(f,f) = R_i B_0^{\frac{1}{2}} \mathcal{A} f^{eq}_{quad}(f,f).$$

Since r_i is, for $i = 1, 2, 3$, in the kernel of $J_0 B^{-\frac{1}{2}}$, we have with suitable coefficients $\alpha_i, \boldsymbol{\beta}_i$

$$r_i = B_0^{\frac{1}{2}} (\alpha_i + \boldsymbol{\beta}_i \cdot \boldsymbol{V}) f^* = B_0^{-\frac{1}{2}} (\alpha_i + \boldsymbol{\beta}_i \cdot \boldsymbol{V}) 1.$$

so that with $g = f^{eq}_{quad}(f,f)$ and the structure of the kernel of $\mathcal{A}$

$$\begin{aligned} P_i J(f) &= \left\langle B_0^{\frac{1}{2}} \mathcal{A} g, B_0^{-\frac{1}{2}} (\alpha_i + \boldsymbol{\beta}_i \cdot \boldsymbol{V}) 1 \right\rangle r_i \\ &= \langle (\alpha_i + \boldsymbol{\beta}_i \cdot \boldsymbol{V}) 1, \mathcal{A} g \rangle = \langle \mathcal{A} (\alpha_i + \boldsymbol{\beta}_i \cdot \boldsymbol{V}) 1, g \rangle = 0. \end{aligned}$$

■

3 Formal asymptotic expansion

In this section, we generally assume that $\mathcal{A}$ is of the form (6) with $\lambda_4 = \lambda_5 = 1/(3\nu) > 0$ for some $\nu > 0$. To investigate the asymptotic behavior of initial value problems for (4) in the limit $\epsilon \to 0$, we introduce a regular expansion $f_\epsilon \sim f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots$ with $f_0 = f^*$. Plugging the expansion into (4) and setting $f_m = 0$ for $m < 0$, we obtain in order ϵ^k, $k \geq -1$

$$\frac{\partial f_k}{\partial t} + \boldsymbol{V} \cdot \nabla f_{k+1} - \mathcal{A}\left(f^{eq}_{lin}(f_{k+2}) - f_{k+2} + \sum_{p+q=k+2} f^{eq}_{quad}(f_p, f_q) \right) = 0, \quad (19)$$

from which we can determine the expansion coefficients f_i. First, we note that in view of (7), (10), and (11),

$$Q(f^{eq}_{lin}(f_{k+2}) - f_{k+2}) = 0, \qquad Q f^{eq}_{quad}(f_n, f_m) = 0.$$

Hence,

$$P(f^{eq}_{lin}(f_{k+2}) - f_{k+2}) = f^{eq}_{lin}(f_{k+2}) - f_{k+2}, \qquad P f^{eq}_{quad}(f_n, f_m) = f^{eq}_{quad}(f_n, f_m),$$

so that an application of the pseudo inverse $\mathcal{A}^\dagger$ to (19), yields in view of (8) for any $k \in \mathbb{Z}$

$$f_k = f^{eq}_{lin}(f_k) + \sum_{p+q=k} f^{eq}_{quad}(f_p, f_q) - \mathcal{A}^\dagger \left(\frac{\partial f_{k-2}}{\partial t} + \boldsymbol{V} \cdot \nabla f_{k-1} \right). \quad (20)$$

We remark that (20) does not specify f_k completely since f_k also appears on the right-hand side as argument of f^{eq}_{lin}. Due to the structure of f^{eq}_{lin}, we can also say that (20) determines f_k up to the moments $\rho_k = \langle 1, f_k \rangle$ and $\boldsymbol{u}_k = \langle 1, \boldsymbol{V} f_k \rangle$. To fix these remaining degrees of freedom, we apply Q to (19) which yields $Q(\partial_t f_k + \boldsymbol{V} \cdot \nabla f_{k+1}) = 0$. In view of (7), we can express this equation also in terms of the moments $\rho_k, \boldsymbol{u}_k$

$$\frac{\partial \rho_k}{\partial t} + \operatorname{div} \boldsymbol{u}_{k+1} = 0, \quad (21)$$

$$\frac{\partial \boldsymbol{u}_k}{\partial t} + \operatorname{div} \langle 1, \boldsymbol{V} \otimes \boldsymbol{V} f_{k+1} \rangle = \boldsymbol{0}. \quad (22)$$

Here, the symmetric tensor product $\boldsymbol{a} \otimes \boldsymbol{b}$ is defined as the matrix with components $(a_i b_j + a_j b_i)/2$ and the divergence is applied row-wise.
In order to carry out the expansion, the following result is crucial.

Lemma 3.1
Assume $\rho_{2m-1}|_{t=0} = 0, \boldsymbol{u}_{2m}|_{t=0} = \boldsymbol{0}$ for $m = 1, 2, \cdots$. Then the expansion coefficients satisfy $S_e f_{2m} = f_{2m}$, and $S_o f_{2m-1} = f_{2m-1}$, i.e. f_{2m} are even functions and f_{2m-1} are odd functions for all $m \in \mathbb{Z}$. The moments ρ_{2m} and $\boldsymbol{u}_{2m-1}$ are solutions of the equation

$$\frac{\partial \boldsymbol{u}_{2m-1}}{\partial t} + c_{2m-1} \operatorname{div} \boldsymbol{u}_{2m-1} \otimes \boldsymbol{u}_1 + \frac{1}{3} \nabla \rho_{2m} = \nu \Delta \boldsymbol{u}_{2m-1} + \boldsymbol{G}_{2m-1} \quad (23)$$

with the divergence condition $\operatorname{div} \boldsymbol{u}_{2m-1} = -\partial_t \rho_{2m-2}$. In the case $m = 1$, we have $c_1 = 1$ and $\boldsymbol{G}_1 = \boldsymbol{0}$. Otherwise, $c_{2m-1} = 2$ and the source terms $\boldsymbol{G}_{2m-1}$, $m > 1$, depend only on derivatives of ρ_k, $\boldsymbol{u}_k$ with $k < 2m - 1$.

Proof: We prove the symmetry result by induction over m, where we add the additional statement $\rho_{2m+1} = 0$ (which follows from $S_o f_{2m-1} = f_{2m-1}$ once the proof is carried out and therefore does need to be stated in the lemma). The induction base $m = 0$ is quite simple because $f_{-1} = 0$ is odd and $f_0 = f^*$ is even. To show that $\rho_1 = 0$, we first exploit relation (20). Taking $k = 1$ and keeping in mind that $f_m = 0$ for $m < 0$ as well as (12) with $f_0 = f^* = S_e f^*$, we conclude $f_1 = f^{eq}_{lin}(f_1)$. In view of (10) and the fact that $\boldsymbol{u}_0 = \langle 1, \boldsymbol{V} f^* \rangle = \boldsymbol{0}$, (22) yields $\nabla \rho_1 = \boldsymbol{0}$ so that ρ_1 is independent of $\boldsymbol{x}$. Integrating (21) over the unit torus Ω, we thus get with the help of the divergence theorem

$$|\Omega| \frac{d\rho_1}{dt} = -\int_\Omega \operatorname{div} \boldsymbol{u}_2 \, d\boldsymbol{x} = 0.$$

Since $\rho_1 = 0$ initially, we conclude that $\rho_1 = 0$ for all $t \geq 0$ which completes the base of induction.
The induction step starts with the observation that f_{2m+1} is odd. This follows from (20) with $k = 2m + 1$ because all terms on the right-hand side are odd functions: $f^{eq}_{lin}(f_{2m+1})$ is odd since $\rho_{2m+1} = 0$ by induction assumption; all quadratic terms $f^{eq}_{quad}(f_p, f_q)$ vanish in view of (12) because if $p + q = 2m + 1$ is odd, either p or q has to be even so that $S_o f_q = 0$ or $S_o f_p = 0$; since f_{2m-1} is odd, the same holds for $\partial_t f_{2m-1}$ and the even symmetry of f_{2m} leads to odd symmetry of $\boldsymbol{V} \cdot \nabla f_{2m}$. The fact that $\mathcal{A}^\dagger$ preserves the symmetry thus shows that f_{2m+1} is odd. Using similar arguments in the case $k = 2m + 2$ (note that $f^{eq}_{quad}(f_p, f_q)$ is even according to (12)), we find that f_{2m+2} is even if and only if $\boldsymbol{u}_{2m+2} = \boldsymbol{0}$. Thus, to finish the induction proof, it remains to show that $\boldsymbol{u}_{2m+2} = \boldsymbol{0}$ and $\rho_{2m+3} = 0$.
Equation (20) with $k = 2m + 3$ and the fact that f_{2n} are even for $n \leq m$ imply

$$f_{2m+3} = f^{eq}_{lin}(f_{2m+3}) + 2 f^{eq}_{quad}(f_{2m+2}, f_1) - \mathcal{A}^\dagger \left(\frac{\partial f_{2m+1}}{\partial t} + \boldsymbol{V} \cdot \nabla f_{2m+2} \right).$$

In view of (22), we multiply this equation with $v_i v_j$ and apply $\langle 1, \cdot \rangle$. Using (10), (11), and summation convention for the repeated indices k, l, we obtain

$$\begin{aligned} \langle 1, V_i V_j f_{2m+3} \rangle = & \frac{\rho_{2m+3}}{3} \delta_{ij} + (\boldsymbol{u}_{2m+2})_i (\boldsymbol{u}_1)_j + (\boldsymbol{u}_{2m+2})_j (\boldsymbol{u}_1)_i \\ & - \frac{\partial (\boldsymbol{u}_{2m+2})_k}{\partial x_l} \left\langle 1, V_i V_j \mathcal{A}^\dagger (3 V_k V_l f^*) \right\rangle. \end{aligned}$$

By direct calculation, one finds

$$\left\langle \mathcal{A}^\dagger V_1^2 1, 3 V_1^2 f^* \right\rangle = \frac{1}{3\lambda_5} = \nu, \qquad \left\langle \mathcal{A}^\dagger V_1^2 1, 3 V_2^2 f^* \right\rangle = -\frac{1}{3\lambda_5} = -\nu,$$

$$\left\langle \mathcal{A}^\dagger V_2^2 1, 3 V_1^2 f^* \right\rangle = -\frac{1}{3\lambda_5} = -\nu, \qquad \left\langle \mathcal{A}^\dagger V_2^2 1, 3 V_2^2 f^* \right\rangle = \frac{1}{3\lambda_5} = \nu,$$

$$\left\langle \mathcal{A}^\dagger V_1 V_2 1, 3 V_1 V_2 f^* \right\rangle = \frac{1}{3\lambda_4} = \nu,$$

and $\left\langle \mathcal{A}^\dagger V_i V_j 1, 3 V_k V_l f^* \right\rangle = 0$ for all other choices of i, j, k and l. We conclude

$$\operatorname{div} \langle 1, \boldsymbol{V} \otimes \boldsymbol{V} f_{2m+3} \rangle = \frac{1}{3} \nabla \rho_{2m+3} + 2 \operatorname{div} \boldsymbol{u}_{2m+2} \otimes \boldsymbol{u}_1 - \nu \Delta \boldsymbol{u}_{2m+2}.$$

Using $\rho_{2m+1} = 0$ from the previous step, we find with (21), (20) that ρ_{2m+3}, $\boldsymbol{u}_{2m+2}$ are obtained as solutions of the Oseen problem

$$\operatorname{div} \boldsymbol{u}_{2m+2} = 0, \qquad \frac{\partial \boldsymbol{u}_{2m+2}}{\partial t} + 2 \operatorname{div} \boldsymbol{u}_{2m+2} \otimes \boldsymbol{u}_1 + \frac{1}{3} \nabla \rho_{2m+3} = \nu \Delta \boldsymbol{u}_{2m+2}.$$

Since the initial data $\rho_{2m+3}|_{t=0}$ and $\boldsymbol{u}_{2m+2}|_{t=0}$ are assumed to be zero, we conclude that $\rho_{2m+3} = 0$, $\boldsymbol{u}_{2m+2} = \mathbf{0}$ is the unique solution of this problem (see Lemma 5.1). This completes the induction.
Finally, let us derive the equation satisfied by $\boldsymbol{u}_{2m-1}$ and ρ_{2m}. The divergence condition is an immediate consequence of (21). To evaluate (22), we note that with (20) applied to $k = 2m$ and $k = 2m-1$

$$f_{2m} = \rho_{2m} f^* + c_{2m-1} f^{eq}_{quad}(f_{2m-1}, f_1) - \frac{\partial (\boldsymbol{u}_{2m-1})_k}{\partial x_l} \mathcal{A}^\dagger (3 V_k V_l f^*) + g_{2m-1}$$

where we have collected all terms involving f_k with $k < 2m-1$ in

$$g_{2m-1} = \sum_{k=2}^{2m-2} f^{eq}_{quad}(f_k, f_{2m-k}) \\ - \mathcal{A}^\dagger \left(\frac{\partial f_{2m-2}}{\partial t} - \boldsymbol{V} \cdot \nabla \mathcal{A}^\dagger \left(\frac{\partial f_{2m-3}}{\partial t} - \boldsymbol{V} \cdot \nabla f_{2m-2} \right) \right).$$

Introducing the field $\boldsymbol{G}_{2m-1} = \operatorname{div} \langle 1, \boldsymbol{V} \otimes \boldsymbol{V} g_{2m-1} \rangle$, the result follows from (22). ■

To determine $\boldsymbol{u}_{2m-1}$ for $m \geq 1$ from (23), appropriate initial conditions are needed. In view of (5), we take

$$\boldsymbol{u}_1(0, \boldsymbol{x}) = \bar{\boldsymbol{u}}(\boldsymbol{x}), \qquad \boldsymbol{u}_{2m-1}(0, \boldsymbol{x}) = \mathbf{0}, \quad m = 2, 3, \cdots. \tag{24}$$

To determine ρ_{2m} for $m \geq 1$ from (23) and thus the expansion, we recall (21) and the periodicity of the data and impose

$$\int_\Omega \rho_{2m}(t, \boldsymbol{x}) \, d\boldsymbol{x} = 0. \tag{25}$$

In addition, we remark that (23) is essentially an Oseen problem for the modified velocity field

$$\tilde{\boldsymbol{u}}_{2m-1} = \boldsymbol{u}_{2m-1} - \nabla \Phi, \qquad \Delta \Phi = -\partial_t \rho_{2m-2}$$

which satisfies the incompressibility condition $\operatorname{div} \tilde{\boldsymbol{u}}_{2m-1} = 0$.
By the above formal process, the expansion f_ϵ can be constructed completely. However, we do not know how to show the convergence of the expansion. Instead, we are interested in truncated expansions of the form

$$f^r_\epsilon = f^* + \epsilon f_1 + \cdots + \epsilon^{r+1} f_{r+1}$$

with r a positive integer and f_k defined by (20) (setting $f_0 = f^*$, $f_p = 0$ for $p < 0$), where the moments $(\boldsymbol{u}_k, \rho_{k+1})$ are either set to zero (if k is even) or taken as solution of equation (23) with (24)-(25) (see the appendix for existence and uniqueness results). Inserting the truncated expansion into (4), we find in order ϵ^k the left-hand side of (19) where now $f_p = 0$ for $p > r+1$. By construction of f_k, this expression vanishes exactly as long as $k+2 \leq r+1$. Thus, f_ϵ^r satisfies

$$\frac{\partial f_\epsilon^r}{\partial t} + \frac{1}{\epsilon}\boldsymbol{V}\cdot\nabla f_\epsilon^r - \frac{1}{\epsilon^2}J(f_\epsilon^r) = \epsilon^r \hat{R}_r \tag{26}$$

with

$$\hat{R}_r = \frac{\partial f_r}{\partial t} + \boldsymbol{V}\cdot\nabla f_{r+1} - \sum_{p+q=r+2} \mathcal{A}f_{quad}^{eq}(f_p, f_q)$$
$$+\,\epsilon\left(\frac{\partial f_{r+1}}{\partial t} - \sum_{p+q=r+3}\mathcal{A}f_{quad}^{eq}(f_p, f_q)\right) - \sum_{k=r+2}^{2r}\epsilon^{k-r}\sum_{p+q=k+2}\mathcal{A}f_{quad}^{eq}(f_p, f_q).$$

The averages $\boldsymbol{u}_\epsilon^r = \langle 1, \boldsymbol{V}f_\epsilon^r\rangle$ and $\rho_\epsilon^r = \langle 1, f_\epsilon^r\rangle$ have expressions (because of even/odd symmetry of the coefficients f_k)

$$\boldsymbol{u}_\epsilon^r = \epsilon\boldsymbol{u}_1 + \epsilon^3\boldsymbol{u}_3 + \ldots, \qquad \rho_\epsilon^r = 1 + \epsilon^2\rho_2 + \epsilon^4\rho_4 + \ldots.$$

Note that, in view of (23) with $m = 1$, $(\boldsymbol{u}_1, \rho_2/3)$ is the solution of the Navier-Stokes equation (3).

To investigate the regularity of the truncated expansion, we introduce some notation related to the Sobolev spaces H^s with s a non-negative integer. $L^2 = H^0$ is the space of square integrable ($\mathcal{F}$- or $\mathcal{F}^2$-valued) functions on the unit torus Ω. Its norm is denoted by $\|\cdot\|$. For $s > 0$, H^s is defined as the space of functions which are in L^2 together with their distributional $\boldsymbol{x}$-derivatives of order $\leq s$. We use $\|\cdot\|_s$ to denote the norm. In addition, we use $C(0,T;H^s)$, $AC(0,T;H^s)$ and $L^1(0,T;H^s)$ to denote the Banach spaces of H^s-valued continuous, (locally if $T = +\infty$) absolutely continuous, and (locally if $T = +\infty$) L^1-integrable functions on the time interval $[0,T]$, respectively.

For simplicity, we consider only the case where $r = 3$. Furthermore, we will often use the following well-known fact (see, e.g., [17]).

Lemma 3.2

Let s_1, s_2 be two non-negative integers and $s_3 = \min\{s_1, s_2, s_1 + s_2 - \sigma_d\} \geq 0$ where $\sigma_d = \lfloor d/2\rfloor + 1 = 2$ for our two-dimensional case $d = 2$. Then the product of functions from H^{s_1} and H^{s_2} is in H^{s_3}, i.e.

$$H^{s_1}H^{s_2} \subset H^{s_3}.$$

where the inclusion symbol $\subset$ indicates the continuity of the embedding.

Lemma 3.3

Assume $s \geq 2$ and $\bar{\boldsymbol{u}} \in H^{s+5}$ with $\operatorname{div}\bar{\boldsymbol{u}} = 0$. Then $f_\epsilon^3 \in C(0,\infty;H^s)$, $\hat{R}_3 \in L^1(0,\infty;H^s)$, and for every $T > 0$, $\sup_{t\leq T}\|\boldsymbol{u}_\epsilon^3(t)\|_s = O(\epsilon)$ and $\int_0^T \|\hat{R}_3(t)\|_s dt = O(1)$ as $\epsilon \to 0$.

Proof: Since $f^3_\epsilon = \sum_{k=0}^4 \epsilon^k f_k$, $\boldsymbol{u}^3_\epsilon = \langle 1, \boldsymbol{V} f^3_\epsilon \rangle$ and

$$\hat{R}_3 = \frac{\partial f_3}{\partial t} + \boldsymbol{V} \cdot \nabla f_4 + \epsilon \left(\frac{\partial f_4}{\partial t} - \mathcal{A}^\dagger f^{eq}_{quad}(f_3, f_3) \right),$$

it suffices to show that

$$\begin{gathered} f_1, f_2 \in C(0,\infty; H^s), \qquad f_3 \in AC(0,\infty; H^s), \\ f_4 \in AC(0,\infty; H^s) \cap L^1(0,\infty; H^{s+1}). \end{gathered} \tag{27}$$

Note that $f_0 = f^*$ is independent of $(t, \boldsymbol{x})$ and thereby in $C(0,\infty; H^s)$. In addition, Lemma 3.2 can be used to show that the quadratic term $f^{eq}_{quad}(f_3, f_3)$ is in $C(0,\infty; H^s)$ if so is f_3. To show (27), we consider the equations for $\boldsymbol{u}_1, \rho_2$, and $\boldsymbol{u}_3, \rho_4$.

Denote by Π the orthogonal projection of L^2 onto its closed subspace consisting of all solenoidal vectors. Then the equations for $(\boldsymbol{u}_1, \rho_2)$ can be rewritten as

$$\begin{aligned} \partial_t \boldsymbol{u}_1 + \Pi(\boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1) = \nu \Delta \boldsymbol{u}_1, \qquad & \Delta \rho_2 = -3 \,\mathrm{div}\,(\boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1), \\ \boldsymbol{u}_1(0, \boldsymbol{x}) = \bar{\boldsymbol{u}}(\boldsymbol{x}), \qquad & \int_\Omega \rho_2(t, \boldsymbol{x})\, d\boldsymbol{x} = 0. \end{aligned} \tag{28}$$

Because $\bar{\boldsymbol{u}} \in H^{s+5}$ with $\mathrm{div}\,\bar{\boldsymbol{u}} = 0$, we deduce easily from the existence theory in [22] for incompressible Navier-Stokes equations (see also the proof of Lemma 5.1) that

$$\begin{aligned} \boldsymbol{u}_1 \in& AC(0,\infty; H^{s+4}) \cap C(0,\infty; H^{s+5}) \cap L^1(0,\infty; H^{s+6}), \\ \rho_2 \in& L^1(0,\infty; H^{s+6}). \end{aligned} \tag{29}$$

This implies that $\boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1 \in AC(0,\infty; H^{s+3})$, since

$$\|\partial_t(\boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1)\|_{s+3} \le C \|\partial_t \boldsymbol{u}_1\|_{s+3} \|\boldsymbol{u}_1\|_{s+4} + C \|\boldsymbol{u}_1\|_{s+3} \|\partial_t \boldsymbol{u}_1\|_{s+4}$$

due to Lemma 3.2. Thus, from the equations in (28) and the familiar fact $\|\rho_2\|_2 \le C \|\Delta \rho_2\|$ we see that

$$\rho_2 \in AC(0,\infty; H^{s+4}) \qquad \text{and} \qquad \partial_t \boldsymbol{u}_1 \in AC(0,\infty; H^{s+2}). \tag{30}$$

Similarly, we have $\partial_t(\boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1) \in AC(0,\infty; H^{s+1})$ and differentiating the equations in (28) with respect to t gives

$$\partial_t \rho_2 \in AC(0,\infty; H^{s+2}) \quad \text{and} \quad \partial_t^2 \boldsymbol{u}_1 \in AC(0,\infty; H^s); \tag{31}$$

moreover, $\partial_t^2(\boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1) \in AC(0,\infty; H^{s-1})$ and

$$\partial_t^2 \rho_2 \in AC(0,\infty; H^s). \tag{32}$$

Now (29) and (30) immediately give

$$\begin{aligned} & f_1 = 3 \boldsymbol{u}_1 \cdot \boldsymbol{V} f^* \in AC(0,\infty; H^{s+4}) \cap L^1(0,\infty; H^{s+6}), \\ & \partial_t f_1 \in AC(0,\infty; H^{s+2}). \end{aligned} \tag{33}$$

Recall that $f_2 = \rho_2 f^* + f^{eq}_{quad}(f_1, f_1) - \mathcal{A}^\dagger(\boldsymbol{V}\cdot\nabla f_1)$. By using Lemma 3.2, it is easy to see from (29) and (30) that

$$\begin{aligned} f^{eq}_{quad}(f_1, f_1) \in & AC(0,\infty; H^{s+4}) \cap C(0,\infty; H^{s+5}), \\ \partial_t f^{eq}_{quad}(f_1, f_1) \in & AC(0,\infty; H^{s+2}). \end{aligned}$$

Thus it follows from (29), (30), (33) and (31) that

$$\begin{aligned} & f_2 \in AC(0,\infty; H^{s+3}) \cap L^1(0,\infty; H^{s+5}), \\ & \partial_t f_2 \in AC(0,\infty; H^{s+1}). \end{aligned} \tag{34}$$

Next we turn to the equations for $\boldsymbol{u}_3$ and ρ_4:

$$\begin{aligned} \operatorname{div}\boldsymbol{u}_3 = -\partial_t\rho_2, \qquad & \partial_t\boldsymbol{u}_3 + 2\operatorname{div}\boldsymbol{u}_3\otimes\boldsymbol{u}_1 + \nabla\rho_4/3 = \nu\Delta\boldsymbol{u}_3 + \boldsymbol{G}_3, \\ \boldsymbol{u}_3(0,\boldsymbol{x}) = 0, \qquad & \int_\Omega \rho_4(t,\boldsymbol{x})\,d\boldsymbol{x} = 0. \end{aligned}$$

Let ϕ be such that $\Delta\phi = -\partial_t\rho_2$ and $\int_\Omega \phi(t,\boldsymbol{x})\,d\boldsymbol{x} = 0$. It follows from (30)-(32) that

$$\phi \in AC(0,\infty; H^{s+4}) \cap L^1(0,\infty; H^{s+6}), \qquad \partial_t\phi \in AC(0,\infty; H^{s+2}).$$

Set $\boldsymbol{w} = \boldsymbol{u}_3 - \nabla\phi$ and $p = \rho_4/3 + \partial_t\phi + \nu\partial_t\rho_2$. Then we have

$$\begin{aligned} \operatorname{div}\boldsymbol{w} = {} & \operatorname{div}\boldsymbol{u}_3 - \Delta\phi = \operatorname{div}\boldsymbol{u}_3 + \partial_t\rho_2 = 0, \\ \partial_t\boldsymbol{w} + 2\operatorname{div}\boldsymbol{w}\otimes\boldsymbol{u}_1 + \nabla p = {} & \nu\Delta\boldsymbol{w} + \boldsymbol{G}_3 - 2\operatorname{div}(\nabla\phi\otimes\boldsymbol{u}_1), \\ \boldsymbol{w}(0,\boldsymbol{x}) = -\nabla\phi(0,\boldsymbol{x}) \in H^{s+3}, \qquad & \int_\Omega p(t,\boldsymbol{x})\,d\boldsymbol{x} = 0. \end{aligned} \tag{35}$$

This is the Oseen problem with an external force $\boldsymbol{h} = \boldsymbol{G}_3 - 2\operatorname{div}(\nabla\phi\otimes\boldsymbol{u}_1)$. Recall that $\boldsymbol{G}_3 = \operatorname{div}\langle 1, \boldsymbol{V}\otimes\boldsymbol{V} g_3\rangle$ with

$$g_3 = -\mathcal{A}^\dagger\left(\frac{\partial f_2}{\partial t} - \boldsymbol{V}\cdot\nabla\mathcal{A}^\dagger\left(\frac{\partial f_1}{\partial t} - \boldsymbol{V}\cdot\nabla f_2\right)\right).$$

We see from (33)-(34) that $\boldsymbol{h} \in AC(0,\infty; H^s) \cap L^1(0,\infty; H^{s+2})$. Thus Lemma 5.1 gives

$$\boldsymbol{w} \in AC(0,\infty; H^{s+1}) \cap L^1(0,\infty; H^{s+3}), \qquad p \in L^1(0,\infty; H^{s+3}).$$

On the other hand, $\boldsymbol{w}\otimes\boldsymbol{u}_1 \in AC(0,\infty; H^{s+1})$ follows from Lemma 3.2 and (29). Taking divergence of (35) gives $\Delta p = \operatorname{div}(\boldsymbol{h} - 2\operatorname{div}\boldsymbol{w}\otimes\boldsymbol{u}_1)$. Thus we also have

$$p \in AC(0,\infty; H^{s+1}).$$

Recall that $\nabla\phi \in AC(0,\infty; H^{s+3}) \cap L^1(0,\infty; H^{s+5})$ and that $\partial_t\phi + \nu\partial_t\rho_2 \in AC(0,\infty; H^{s+2}) \cap L^1(0,\infty; H^{s+4})$. Then we have

$$\boldsymbol{u}_3 = \boldsymbol{w} + \nabla\phi, \quad \rho_4/3 = p - (\partial_t\phi + \nu\partial_t\rho_2) \in AC(0,\infty; H^{s+1}) \cap L^1(0,\infty; H^{s+3}).$$

Together with (33) and (34), this gives

$$f_3 = 3\boldsymbol{u}_3 \cdot \boldsymbol{V} f^* - \mathcal{A}^\dagger \left(\frac{\partial f_1}{\partial t} + \boldsymbol{V} \cdot \nabla f_2 \right) \in AC(0,\infty; H^{s+1}) \cap L^1(0,\infty; H^{s+3}),$$

$$f_4 = \rho_4 f^* + 2 f^{eq}_{quad}(f_1, f_3) - \mathcal{A}^\dagger \left(\frac{\partial f_2}{\partial t} + \boldsymbol{V} \cdot \nabla f_3 \right) \in AC(0,\infty; H^{s}) \cap L^1(0,\infty; H^{s+2}).$$

Hence (27) is verified. ■

We conclude this section with a more detailed description of f^3_ϵ.

Lemma 3.4
The truncated expansion f^3_ϵ coincides up to terms of order ϵ^3 with the Chapman-Enskog distribution $F_{CE}(p, \boldsymbol{u})$ corresponding to the solution $(\boldsymbol{u}, p)$ of the Navier Stokes equation (3),

$$F_{CE}(p, \boldsymbol{u}) = F^{eq}_1(1, \epsilon \boldsymbol{u}) + \epsilon^2 \left(3p - \frac{9}{2} \nu S[\boldsymbol{u}] : (\boldsymbol{V} \otimes \boldsymbol{V} - |\boldsymbol{V}|^2/2) \right) f^*,$$

where $S_{ij}[\boldsymbol{u}] = \partial_{x_j} u_i + \partial_{x_i} u_j$ is the viscous stress tensor and : denotes the matrix scalar product $A : B = \sum_{ij} A_{ij} B_{ij}$.

Proof: According to our construction, $f^3_\epsilon = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \mathcal{O}(\epsilon^3)$ with

$$f_0 = f^*, \quad f_1 = 3\boldsymbol{u} \cdot \boldsymbol{V} f^*, \quad f_2 = 3p f^* + f^{eq}_{quad}(f_1, f_1) - \mathcal{A}^\dagger(\boldsymbol{V} \cdot \nabla f_1).$$

Since $f_0 + \epsilon f_1 = F^{eq}_{lin}(1, \epsilon \boldsymbol{u})$ and $\epsilon^2 f^{eq}_{quad}(f_1, f_1) = F^{eq}_{quad}(\epsilon \boldsymbol{u}, \epsilon \boldsymbol{u})$, we thus have

$$f^3_\epsilon = F^{eq}_1(1, \epsilon \boldsymbol{u}) + \epsilon^2 (3p f^* - \mathcal{A}^\dagger(\boldsymbol{V} \cdot \nabla f_1) + \mathcal{O}(\epsilon^3).$$

An explicit calculation of $\mathcal{A}^\dagger(\boldsymbol{V} \cdot \nabla f_1)$ yields

$$\mathcal{A}^\dagger(\boldsymbol{V} \cdot \nabla f_1) = \nabla \boldsymbol{u} : \mathcal{A}^\dagger(3\boldsymbol{V} \otimes \boldsymbol{V} f^*) = 9\nu \nabla \boldsymbol{u} : (\boldsymbol{V} \otimes \boldsymbol{V} - |\boldsymbol{V}|^2/2) f^*.$$

Since $\boldsymbol{v} \otimes \boldsymbol{v} - |\boldsymbol{v}|^2/2$ is a symmetric matrix, we can replace the Jacobian $\nabla \boldsymbol{u}$ also by its symmetric part $(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)/2 = S[\boldsymbol{u}]/2$ without changing the matrix scalar product. This completes the proof. ■

4 Justification of formal approximations

Having constructed formal asymptotic approximations f^r_ϵ for initial-value problems of (4), we prove in this section the validity of the approximations. The main result is

Theorem 4.1
Suppose $s \geq 2$ is an integer, $\bar{\boldsymbol{u}} \in H^{s+5}$ with $\operatorname{div} \bar{\boldsymbol{u}} = 0$, $\bar{p}$ is the solution of $\Delta p = -\operatorname{div}(\bar{\boldsymbol{u}} \cdot \nabla \bar{\boldsymbol{u}})$ and $\int_\Omega p\, d\boldsymbol{x} = 0$, and $T_0 > 0$ is any given finite number. Then the lattice Boltzmann model (4), with $\mathcal{A}$ of the form (6), $\lambda_4 = \lambda_5 = 1/(3\nu)$, $\lambda_6 = \lambda_9$, and initial data

$$f^\epsilon|_{t=0} = F_1^{eq}(1, \epsilon\bar{\boldsymbol{u}}) + \epsilon^2 \left(3\bar{p} - \frac{9}{2}\nu S[\bar{\boldsymbol{u}}] : (\boldsymbol{V} \otimes \boldsymbol{V} - |\boldsymbol{V}|^2/2)\right) f^*$$

has a unique solution $f^\epsilon \in C(0, T_0; H^s)$. Moreover, there exist $\epsilon_0 = \epsilon_0(T_0) > 0$ and $K = K(T_0) > 0$ such that for all positive $\epsilon < \epsilon_0$

$$\|f_\epsilon^3(t) - f^\epsilon(t)\|_s \leq K\epsilon^3, \qquad t \in [0, T_0].$$

In particular, the velocity field $\boldsymbol{u}_\epsilon/\epsilon = \langle 1, \boldsymbol{V} f^\epsilon \rangle / \epsilon$ coincides with the solution $\boldsymbol{u}$ of the Navier-Stokes equation (3) up to order ϵ^2 and $(\langle 1, f^\epsilon \rangle - 1)/(3\epsilon^2)$ with the pressure p up to $O(\epsilon)$:

$$\|\boldsymbol{u} - \boldsymbol{u}_\epsilon/\epsilon\|_s \leq K\epsilon^2, \qquad \|p - (\langle 1, f^\epsilon \rangle - 1)/(3\epsilon^2)\|_s \leq K\epsilon.$$

The proof of Theorem 4.1.
Since $f^\epsilon(0, \cdot) \in H^s$ with $s > d/2 = 1$, by the local existence theory for IVPs of symmetrizable hyperbolic systems (see [17]), there is a time interval $[0, T]$ so that (4) has a unique H^s-solution

$$f^\epsilon \in C([0, T], H^s).$$

Define

$$T_\epsilon = \sup\left\{T > 0 : f^\epsilon \in C([0, T], H^s)\right\}. \tag{36}$$

Namely, $[0, T_\epsilon)$ is the maximal time interval of H^s existence. Thanks to the *convergence-stability lemma* in [24, 25], it suffice to prove the error estimate for $t \in [0, \min\{T_0, T_\epsilon\})$. Indeed, once the estimate is proved, the lemma can be used to show $T_\epsilon > T_0$.
To this end, we compute from equations (4) and (26) that the error $E = f_\epsilon^3 - f^\epsilon$ satisfies

$$\frac{\partial E}{\partial t} + \frac{1}{\epsilon}\boldsymbol{V} \cdot \nabla E = \frac{J(f_\epsilon^3) - J(f^\epsilon)}{\epsilon^2} + \epsilon^3 \hat{R}_3.$$

We differentiate this equation with ∇^α (in $\boldsymbol{x}$) for a multi-index α satisfying $|\alpha| \leq s$ to get with $E_\alpha = \nabla^\alpha E$

$$\frac{\partial E_\alpha}{\partial t} + \frac{1}{\epsilon}\boldsymbol{V} \cdot \nabla E_\alpha = \frac{1}{\epsilon^2} J_0 E_\alpha + F_\alpha + H_\alpha, \tag{37}$$

where

$$F_\alpha = \frac{1}{\epsilon^2}\nabla^\alpha \left(J(f_\epsilon^3) - J(f^\epsilon) - J_0 E\right), \qquad H_\alpha = \epsilon^3 \nabla^\alpha \hat{R}_3$$

For the sake of clarity, we divide the following arguments into lemmas.

Lemma 4.2
Under the conditions of Theorem 4.1, we have

$$\frac{d}{dt}\int_\Omega \langle B_0E_\alpha, E_\alpha\rangle\, d\boldsymbol{x} + C\frac{\|P_{II}E_\alpha\|^2}{\epsilon^2} \le C\epsilon^3\|E_\alpha\|\|\nabla^\alpha \hat{R}_3\| + C\epsilon^2\|F_\alpha\|^2.$$

Here $P_{II} = \sum_{k=4}^{9} P_k$, and C denotes a generic constant.

Proof: Applying B_0 to equation (37) as well as $\langle\cdot, E_\alpha\rangle$ we find (using the fact that B_0 and B_0V_j are multiplication operators and thus self-adjoint)

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\langle B_0E_\alpha, E_\alpha\rangle + \frac{1}{2\epsilon}\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\langle B_0V_jE_\alpha, E_\alpha\rangle\\ =&\frac{1}{\epsilon^2}\langle B_0J_0E_\alpha, E_\alpha\rangle + \langle B_0F_\alpha, E_\alpha\rangle + \langle B_0H_\alpha, E_\alpha\rangle .\end{aligned} \tag{38}$$

Thanks to relations (16) and the fact that $\lambda_1, \lambda_2, \lambda_3 = 0$, it follows that

$$\langle B_0J_0E_\alpha, E_\alpha\rangle = -\sum_{k=4}^{9}\lambda_k\langle P_kE_\alpha, P_kE_\alpha\rangle \le -\lambda_{min}\langle P_{II}E_\alpha, P_{II}E_\alpha\rangle$$

where $\lambda_{min} = \min\{\lambda_k : k = 4, \ldots, 9\}$. Setting $P_I = \sum_{k=1}^{3} P_k$, (17) implies $P_IF_\alpha \equiv 0$. Thanks to (16), we have

$$\langle B_0F_\alpha, E_\alpha\rangle = \sum_{k=4}^{9}\langle P_kF_\alpha, P_kE_\alpha\rangle \le \frac{\lambda_{min}}{2}\frac{\langle P_{II}E_\alpha, P_{II}E_\alpha\rangle}{\epsilon^2} + C\epsilon^2\langle F_\alpha, F_\alpha\rangle . \tag{39}$$

Finally,

$$\langle B_0H_\alpha, E_\alpha\rangle \le C|E_\alpha||H_\alpha|.$$

Thus, integrating (38) with respect to $\boldsymbol{x}$ over Ω, the result follows. ■

The next Lemma is used to estimate F_α.

Lemma 4.3
Set $\triangle(t) = \|E(t)\|_s/\epsilon = \|f_\epsilon^3(t) - f^\epsilon(t)\|_s/\epsilon$. Then we have under the conditions of Theorem 4.1 for $|\alpha| \le s$

$$\epsilon\|F_\alpha(t)\| \le C(1 + \triangle(t))\|E(t)\|_s. \tag{40}$$

Proof: Observe

$$J(f_\epsilon^3) - J(f^\epsilon) - J_0E = \int_0^1 \big(DJ(f(\theta)) - J_0\big)E\, d\theta$$

with $f(\theta) = f_\epsilon^3 + (1-\theta)(f^\epsilon - f_\epsilon^3)$. From (15), the definition of F_{quad}^{eq}, and Lemma 3.2, it follows that

$$\|(DJ(f(\theta)) - J_0)E\|_s \le C\|\boldsymbol{u}(\theta)\|_s\|E\|_s, \qquad \boldsymbol{u}(\theta) = \langle 1, \boldsymbol{V} f(\theta)\rangle$$

Since $\|\boldsymbol{u}(\theta)\|_s \le \|\boldsymbol{u}_\epsilon^3\|_s + \|\boldsymbol{u}^\epsilon - \boldsymbol{u}_\epsilon^3\|_s \le C\epsilon + C\epsilon\triangle$, we obtain

$$\|\nabla^\alpha \left(J(f_\epsilon^3) - J(f^\epsilon) - J_0 E\right)\| \le C\int_0^1 \|\boldsymbol{u}(\theta)\|_s \, d\theta \, \|E\|_s \le C\epsilon(1+\triangle)\|E\|_s.$$

Hence $\|F_\alpha\| \le \frac{C(1+\triangle)}{\epsilon}\|E\|_s$ and (40) follows. ■

Substituting (40) into the inequality in Lemma 4.2 yields

$$\frac{d}{dt}\int_\Omega \langle B_0 E_\alpha, E_\alpha\rangle \, d\boldsymbol{x} \le C\epsilon^3\|E_\alpha\|\|\nabla^\alpha \hat{R}_3\| + C(1+\triangle^2)\|E\|_s^2. \qquad (41)$$

Note that $C^{-1}\langle E_\alpha, E_\alpha\rangle \le \langle B_0 E_\alpha, E_\alpha\rangle \le C\langle E_\alpha, E_\alpha\rangle$. We integrate (41) from 0 to T with $[0,T] \subset [0, \min\{T_\epsilon, T_0\})$ to obtain

$$\|E_\alpha(T)\|^2 \le C\epsilon^6 + C\epsilon^3\int_0^T \|E_\alpha(t)\|\|\nabla^\alpha \hat{R}_3(t)\|dt + C\int_0^T \left(1+\triangle^2\right)\|E(t)\|_s^2 dt.$$

Here we have used $\|E(0)\|_s = O(\epsilon^3)$. Summing up this inequality for the multi-index α with $|\alpha| \le s$, we get

$$\|E(T)\|_s^2 \le C\epsilon^6 + C\epsilon^3\int_0^T \|E(t)\|_s\|\hat{R}_3(t)\|_s dt + C\int_0^T \left(1+\triangle^2\right)\|E(t)\|_s^2 dt.$$

Denote by $F(T)$ the *square root* of the right-hand side of the last inequality. We have $\|E(t)\|_s \le F(t), F(0) = O(\epsilon^3)$ and

$$F(t)F'(t) = C\epsilon^3\|E(t)\|_s\|\hat{R}_3(t)\|_s + C(1+\triangle^2)\|E(t)\|_s^2.$$

Moreover, we have

$$F'(t) \le C\epsilon^3\|\hat{R}_3(t)\|_s + C(1+\triangle^2)F(t). \qquad (42)$$

Recall from Lemma 3.3 that $\int_0^{T_0}\|\hat{R}_3(t)\|_s dt = O(1)$. We apply Gronwall's lemma to (42) to obtain

$$F(T) \le C\epsilon^3 \exp\left[C\int_0^T \left(1+\triangle^2\right)dt\right]. \qquad (43)$$

Since $F \ge \|E\|_s = \epsilon\triangle$, it follows from (43) that

$$\triangle(T) \le C\epsilon^2 \exp\left[C\int_0^T \left(1+\triangle^2\right)dt\right] \equiv \Phi(T). \qquad (44)$$

Thus,

$$\Phi'(t) = C(1+\triangle^2)\Phi(t) \le C\Phi(t) + C\Phi^3(t).$$

Applying the nonlinear Gronwall-type inequality in [23] to the last inequality yields

$$\Phi(t) \le \exp(CT_0),$$

for $t \in [0, \min\{T_\epsilon, T_0\})$ if we choose ϵ so small that

$$\Phi(0) = C\epsilon^2 \le e^{-CT_0}.$$

Because of (44), there exists a constant c, independent of ϵ, such that

$$\triangle(T) \le c \tag{45}$$

for any $T \in [0, \min\{T_\epsilon, T_0\})$. Finally, the theorem is concluded from (43) with (45) and $\|E\|_s \le F$. This completes the proof of the theorem 4.1.

5 Appendix

Here we slightly modify a proof in [16] to formulate an existence theorem for the Oseen problem

$$\begin{aligned} \operatorname{div} \boldsymbol{u} = 0, \qquad & \frac{\partial \boldsymbol{u}}{\partial t} + \operatorname{div} \boldsymbol{u} \otimes \boldsymbol{u}_1 + \nabla p = \nu \Delta \boldsymbol{u} + \boldsymbol{h}, \\ \boldsymbol{u}(0, \boldsymbol{x}) = \bar{\boldsymbol{u}}(\boldsymbol{x}), \qquad & \int_\Omega p(t, \boldsymbol{x})\, d\boldsymbol{x} = 0. \end{aligned} \tag{46}$$

in a *periodic* domain. Here $\boldsymbol{u}_1, \boldsymbol{h}, \bar{\boldsymbol{u}}$ are given functions which are periodic in $\boldsymbol{x}$, and $\operatorname{div} \bar{\boldsymbol{u}} = 0$.

Lemma 5.1
Let $m \ge 0$ be an integer and $T > 0$ a real number. Assume $\boldsymbol{u}_1 \in C(0, T; H^{s+1})$ with $s \ge \max\{m, \sigma_d\}$, $\boldsymbol{h} \in L^1(0, T; H^m)$ and $\bar{\boldsymbol{u}} \in H^m$ with $\operatorname{div} \bar{\boldsymbol{u}} = 0$. Then the Oseen problem (46) has a unique solution $(\boldsymbol{u}, p)$ satisfying

$$\begin{aligned} \boldsymbol{u} \in & AC(0, T; H^{m-1}) \cap C(0, T; H^m) \cap L^1(0, T; H^{m+1}), \\ p \in & L^1(0, T; H^{m+1}). \end{aligned}$$

Proof: Denote by H^m_σ the closed subspace of H^m consisting of all solenoidal vectors. We decouple (46) as

$$\begin{aligned} \frac{d\boldsymbol{u}}{dt} + A\boldsymbol{u} = \boldsymbol{F}(\boldsymbol{u}, \boldsymbol{u}_1) + \Pi \boldsymbol{h}(t), \qquad & \boldsymbol{u}(0, \boldsymbol{x}) = \bar{\boldsymbol{u}}(\boldsymbol{x}), \\ \Delta p = \operatorname{div}(\boldsymbol{h} - \operatorname{div} \boldsymbol{u} \otimes \boldsymbol{u}_1), \qquad & \int_\Omega p(t, \boldsymbol{x})\, d\boldsymbol{x} = 0. \end{aligned} \tag{47}$$

Here $A = -\nu\Delta$ is a nonnegative self-adjoint operator in H_σ; Π is the orthogonal projection of H^m onto H^m_σ; and $\boldsymbol{F}$ is a bilinear operator

$$\boldsymbol{F}(\boldsymbol{u}, \boldsymbol{u}_1) = -\Pi \operatorname{div} \boldsymbol{u} \otimes \boldsymbol{u}_1.$$

We firstly show the existence of $\boldsymbol{u}$. According to [16], it suffices to construct a solution to the integral equation

$$\boldsymbol{u}(t) \equiv G\boldsymbol{u}(t) = e^{-tA}\bar{\boldsymbol{u}} + \int_0^t e^{-(t-s)A}[\boldsymbol{F}(\boldsymbol{u}, \boldsymbol{u}_1) + \Pi \boldsymbol{h}]ds. \tag{48}$$

To this end we shall use the method of contraction map.
For simplicity we write $X_m = C(0, T'; H^m_\sigma), Y_m = L^1(0, T'; H^m_\sigma)$ and set $Z = X_m \cap Y_{m+1}$. Here $T' > 0$ is to be determined later. For the norm in Z we choose

$$\|\boldsymbol{w}\|_Z = \max\{\|\boldsymbol{w}\|_{X_m}, L^{-1}\|\boldsymbol{w}\|_{Y_{m+1}}\},$$

where $L > 0$ is also to be determined later.
Since Π has norm one in any H^m, we use Lemma 3.2 to obtain

$$\begin{aligned}
\|\boldsymbol{F}(\boldsymbol{u}, \boldsymbol{u}_1)\|_m \leq& \|\operatorname{div} \boldsymbol{u} \otimes \boldsymbol{u}_1\|_m \\
\leq& C(\|\boldsymbol{u}\|_m \|\nabla \boldsymbol{u}_1\|_s + \|\boldsymbol{u}_1\|_s \|\nabla \boldsymbol{u}\|_m) \\
\leq& C\|\boldsymbol{u}\|_{m+1}\|\boldsymbol{u}_1\|_{s+1} = C\|\boldsymbol{u}\|_{m+1}.
\end{aligned}$$

Thus, we have

$$\|\boldsymbol{F}(\boldsymbol{u}, \boldsymbol{u}_1)\|_{Y_m} \leq C\|\boldsymbol{u}\|_{Y_{m+1}}. \tag{49}$$

Next we compute $G\boldsymbol{u} - G\boldsymbol{w}$ for $\boldsymbol{u}, \boldsymbol{w} \in Z$. We have

$$G\boldsymbol{u}(t) - G\boldsymbol{w}(t) = \int_0^t e^{-(t-s)A} F(\boldsymbol{u} - \boldsymbol{w}, \boldsymbol{u}_1)ds.$$

Since e^{-tA} has norm one as an operator in H^m, we conclude

$$\|G\boldsymbol{u} - G\boldsymbol{w}\|_{X_m} \leq \|\boldsymbol{F}(\boldsymbol{u} - \boldsymbol{w}, \boldsymbol{u}_1)\|_{Y_m} \leq C\|\boldsymbol{u} - \boldsymbol{w}\|_{Y_{m+1}}. \tag{50}$$

Since e^{-tA} has norm $(\pi t)^{-1/2}$ as an operator from H^m to H^{m+1}, it follows that $\|G\boldsymbol{u} - G\boldsymbol{w}\|_{m+1}$ is majorized by the convolution of $\|\boldsymbol{F}(\boldsymbol{u} - \boldsymbol{w}, \boldsymbol{u}_1)\|_m$ and $(\pi t)^{-1/2}$. Hence

$$\|G\boldsymbol{u} - G\boldsymbol{w}\|_{Y_{m+1}} \leq 2(T'/\pi)^{1/2} C\|\boldsymbol{u} - \boldsymbol{w}\|_{Y_{m+1}}. \tag{51}$$

We now take $L = 2(T'/\pi)^{1/2}$. Recalling the definition of $\|\cdot\|_Z$ and comparing (50) and (51), we thus obtain

$$\|G\boldsymbol{u} - G\boldsymbol{w}\|_Z \leq CL\|\boldsymbol{u} - \boldsymbol{w}\|_Z. \tag{52}$$

A similar (and simpler) computation gives

$$\|G\boldsymbol{0}\|_Z \leq B \equiv \|\bar{\boldsymbol{u}}\|_m + \int_0^{T'} \|\boldsymbol{h}(t)\|_m dt.$$

These results show that G maps Z into itself. Moreover, if T' is sufficiently small, we have $CL < 1$.

With such choices of T' and L, G maps Z into Z. At the same time, we see from (52) that G is a strict contraction map on Z. Therefore G has a unique fixed point $\boldsymbol{u}$ in Z, which is a local solution of the integral equation (48).
Since T' depends only on $\|\boldsymbol{u}_1\|_{s+1}$, the solution can be directly extended to $[0, T]$. Moreover, from the equation in (47) and the estimate in (49) we see that $\boldsymbol{u} \in AC(0, T; H^{m-1})$.
Finally, we turn to the Poisson equation in (47) for p. Since $\|\nabla p\|$ is a norm equivalent to $\|p\|_1$ in the closed subspace $S \subset H^1$:

$$S := \left\{ p \in H^1 : \int_\Omega p(\boldsymbol{x})\, d\boldsymbol{x} = 0 \right\},$$

the Poisson equation has a unique solution $p \in S$ satisfying

$$\|p\|_{m+1} \leq C \| \operatorname{div} (\boldsymbol{h} - \operatorname{div} \boldsymbol{u} \otimes \boldsymbol{u}_1)\|_{m-1} \leq C(\|\boldsymbol{h}\|_m + \|\boldsymbol{u}_1\|_{s+1} \|\boldsymbol{u}\|_{m+1}).$$

Therefore p is in $L^1(0, T; H^{m+1})$. ∎

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