

# A Finite Difference Interpretation of the Lattice Boltzmann Method

Michael Junk\*

## Abstract

Compared to conventional techniques in computational fluid dynamics, the lattice Boltzmann method (LBM) seems to be a completely different approach to solve the incompressible Navier-Stokes equation. The aim of this article is to correct this impression by showing the close relation of LBM to two standard methods: relaxation schemes and explicit finite difference discretizations. As a side effect, new starting points for a discretization of the incompressible Navier-Stokes equation are obtained.

**Keywords.** discrete velocity models, lattice Boltzmann method, low Mach number limit, incompressible Navier-Stokes equation, finite difference method, relaxation systems, pseudo-compressibility methods

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## 1 Introduction

In recent years, the lattice Boltzmann method has been proposed as a potential alternative to conventional methods in computational fluid dynamics. The basic idea of LBM is to use a very simple microscopic model of a gas which is nevertheless capable of correctly describing the macroscopic flow behavior. The microscopic approach of LBM has its origins in the theory of lattice gas automata (see [1]) and is closely related to discrete velocity models of the Boltzmann equation [2]. Essentially, the velocities of the gas particles are restricted in such a way, that microscopic movement can only take place between the nodes of a regular space lattice. However, since the density of particles per direction can vary continuously, the average velocity of the gas is not discrete. If the collision process among the gas particles is modeled appropriately and if the system is in a particular asymptotic situation (diffusion

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\*FB Mathematik, Universität Kaiserslautern, Erwin-Schrödingerstraße, 67663 Kaiserslautern, Germany,(junk@mathematik.uni-kl.de).

limit), the average velocity approximates a solution of the incompressible Navier-Stokes equation.

In its standard form, LBM is an explicit finite difference approximation of a velocity-discrete Boltzmann equation with a collision operator of relaxation type. The variables in the kinetic description are the particle densities per discrete velocity. They always outnumber the macroscopic flow variables which are obtained as averages based on the particle densities (see [3, 4] for detailed reviews and references on LBM).

Since LBM reduces complicated macroscopic phenomena to a simple microscopic dynamics, it is an interesting object for many researchers. However, if LBM is considered mainly as a numerical method for macroscopic equations, like the incompressible Navier-Stokes system, it is natural to ask for its relation to already existing schemes. The answer to this question is complicated by the fact that LBM is formulated in kinetic terms and not directly in terms of the target equations. Nevertheless, the closeness of LBM to finite difference methods has already been mentioned. For example, in the interesting article [5], lattice Boltzmann methods are presented as subclass of so called *fully Lagrangian schemes* which are shown to be directly related to standard finite difference methods for some particular cases.

In the present article, we investigate the frequently used lattice Boltzmann model for two-dimensional Navier-Stokes flow based on nine discrete velocities and a regular square lattice with the aim to show the close relation to standard methods. It turns out that the lattice Boltzmann algorithm can be viewed as a non-standard way of writing an explicit finite difference approximation of either the Navier-Stokes equation directly or of some relaxation system for the Navier-Stokes equation. The close relation between the kinetic approach of LBM and finite difference methods rests on the basic observation presented in Section 5: discrete microscopic transport plus velocity averaging is equivalent to a finite difference approximation. Since transport and averaging are always ingredients of LBM, the observations in Section 5 can be used to translate other lattice Boltzmann methods into finite difference schemes so that the presented results are not restricted to the example under consideration.

In contrast to the standard treatment, which is based on a two-scale Chapman-Enskog expansion to relate LBM and Navier-Stokes equation, we use the diffusion scaling of the Boltzmann equation (see Section 3) which seems to be a simpler approach.

In Section 6, it is shown that for a special choice of the collision parameter in LBM, the method can be rewritten as explicit finite difference approximation of a compressible Navier-Stokes system (which reduces to the incompressible case in low Mach number flows). In this reformulation, all aspects of the kinetic approach have disappeared or, more precisely, are condensed in the structure of the finite difference stencils for the differential operators in the compressible Navier-Stokes equation. As a common feature, these stencils involve diagonal neighbors and can

be viewed as convex combinations of usual central differences. Despite the unusual size of the stencils, their evaluation is very efficient in the original lattice-Boltzmann formulation. We also find, that the scheme contains no special treatment of the stiff velocity-pressure coupling which arises in nearly incompressible situations. In fact, LBM in its standard form works in the stability constellation  $\Delta t/\Delta x^2 = \mathcal{O}(1)$  known from the explicit scheme for advection diffusion equations. The approximate divergence-free condition is assured by a pseudo-compressibility approach which contains a pressure stabilization, Chorin's idea of artificial compressibility, as well as convective terms which are usually not considered.

For choices of the collision parameter different from the one in Section 6, the situation is slightly more complicated and leads to an interpretation of LBM as superposition of finite difference schemes (Section 7) or as finite difference approximation of a relaxation system for the incompressible Navier-Stokes equation (Section 8).

It is a general observation that an interpretation of a new method in terms of other approaches can have two advantages: First, the new method is better understood and further research can take advantage from the knowledge already available. Secondly, new developments can be triggered in already established fields. In this respect, it is important to note that LBM leads to stable and reliable results so that it is worthwhile to investigate the mechanisms leading to this behavior.

## 2 The Lattice Boltzmann Method

The basic kinetic model is given by the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\nabla f = J(f) \quad (1)$$

which describes the evolution of a particle density  $f(\mathbf{x}, \mathbf{v}, t)$ . The left hand side of (1) represents free transport of the particles while the right hand side describes interactions through collisions. Continuous and discrete velocity models only differ in the structure of the phase space  $\mathcal{X} \times \mathcal{V}$ . In the classical Boltzmann equation, the velocity domain  $\mathcal{V}$  is the full space  $\mathbb{R}^d$ , while discrete models are based on

$$\mathcal{V} = \{\mathbf{c}_0, \dots, \mathbf{c}_{N-1}\} \quad \mathbf{c}_i \in \mathbb{R}^d.$$

The space part  $\mathcal{X}$  is continuous in both cases. In the following, we will consider a two-dimensional model ( $d = 2$ ) with nine velocities,  $\mathbf{c}_0 = \mathbf{0}$  and

$$\begin{array}{cccc} \mathbf{c}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \mathbf{c}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \mathbf{c}_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} & \mathbf{c}_4 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ \mathbf{c}_5 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \mathbf{c}_6 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} & \mathbf{c}_7 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} & \mathbf{c}_8 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{array}$$

which point to the corners and edges of the unit square. However, our observations are not limited to this special situation and we will formulate most steps in a general notation.

We start by introducing the discrete velocity integral for functions  $\psi : \mathcal{V} \mapsto \mathbb{R}$

$$\langle \psi \rangle = \sum_{i=0}^{N-1} \psi(\mathbf{c}_i).$$

Note that  $\psi$  can be identified with the  $N$ -vector of its values  $\psi(\mathbf{c}_0), \dots, \psi(\mathbf{c}_{N-1})$ . For particle distributions  $f(\mathbf{x}, \mathbf{v}, t)$  we therefore introduce

$$f_i(\mathbf{x}, t) = f(\mathbf{x}, \mathbf{c}_i, t) \quad i = 0, \dots, N - 1.$$

Macroscopic quantities like mass density  $\rho$  and momentum density  $\rho \mathbf{u}$  are obtained by taking velocity moments of  $f$

$$\rho(\mathbf{x}, t) = \langle f(\mathbf{x}, \mathbf{v}, t) \rangle, \quad \rho \mathbf{u}(\mathbf{x}, t) = \langle \mathbf{v} f(\mathbf{x}, \mathbf{v}, t) \rangle \quad (2)$$

In many lattice Boltzmann applications, the collision operator  $J(f)$  in (1) is of BGK-type

$$J(f) = -\frac{1}{\tau}(f - f^{eq}(\rho, \mathbf{u})). \quad (3)$$

where  $\tau > 0$  is called *relaxation time* and  $f^{eq}$  is the *equilibrium distribution*, which depends on  $f$  through the parameters  $\rho$  and  $\mathbf{u}$  in (2). Other models are based on

$$J(f)(\mathbf{w}) = \langle \mathcal{A}(\mathbf{w}, \mathbf{v})(f - f^{eq})(\mathbf{v}) \rangle$$

which can be viewed as linearizations of general nonlinear collision operators [6, 7]. Conservation of mass and momentum in the collision process translates into

$$\langle J(f) \rangle = 0 \quad \text{and} \quad \langle \mathbf{v} J(f) \rangle = \mathbf{0}$$

which puts additional conditions on the kernel  $\mathcal{A}$ . In (3), it is a direct consequence of the construction, since  $f$  and  $f^{eq}$  have identical mass and momentum densities. In the following, we assume the simple structure (3) but the considerations can also be applied to collision matrices.

In the standard D2Q9-model [8], the equilibrium distribution has the form

$$f^{eq}(\rho, \mathbf{u}; \mathbf{v}) = \rho \left( 1 + 3u_k v_k + \frac{9}{2} u_k u_l (v_k v_l - \delta_{kl}/3) \right) f^*(\mathbf{v}) \quad (4)$$

where Einstein's summation convention is used for the indices  $k, l$  (taking values 1, 2) and  $f^*$  is defined by

$$f^*(\mathbf{c}_i) = \begin{cases} \frac{4}{9} & i = 0 \\ \frac{1}{9} & i = 1, \dots, 4 \\ \frac{1}{36} & i = 5, \dots, 8 \end{cases} \quad (5)$$

The actual lattice Boltzmann evolution is obtained from (1) by a discretization in space and time. First, (1) is split into a collision step  $\partial_t f = J(f)$  and a free flow

step  $\partial_t f + \mathbf{v}\nabla f = 0$ . Then, the time derivative in the collision part is approximated by an explicit Euler step

$$\tilde{f}(\mathbf{x}, \mathbf{v}, t + \delta t) = f(\mathbf{x}, \mathbf{v}, t) + \delta t J(f)(\mathbf{x}, \mathbf{v}, t).$$

Finally, the free flow equation is solved explicitly with  $\tilde{f}(\mathbf{x}, \mathbf{v}, t + \delta t)$  as initial value, giving rise to the lattice Boltzmann evolution

$$f(\mathbf{x} + \mathbf{v}\delta t, \mathbf{v}, t + \delta t) = f(\mathbf{x}, \mathbf{v}, t) - \frac{\delta t}{\tau}(f - f^{eq})(\mathbf{x}, \mathbf{v}, t). \quad (6)$$

The space discretization is obtained by simply restricting  $\mathbf{x}$  to the nodes of a regular square lattice with side length  $\delta x = |\mathbf{c}_1|\delta t$ . Due to the structure of the discrete velocities  $\mathbf{c}_0, \dots, \mathbf{c}_8$ , the shifted positions  $\mathbf{x} + \mathbf{v}\delta t$  are then automatically nodes of the same grid and (6) is completely discretized. Apart from this regular discretization, which makes (6) a finite difference approximation of (1), other approaches including curvilinear and unstructured meshes, have been discussed (see [9, 10, 11]).

If the simple updating rule (6) is used iteratively, while obeying certain conditions which we detail in the next section, it turns out that  $\mathbf{u} = \langle \mathbf{v}f \rangle / \langle f \rangle$  gives rise to an approximate solution of the incompressible Navier-Stokes equation with Reynolds number related to the relaxation parameter  $\tau$

$$\frac{1}{Re} = \frac{1}{3} \left( \frac{\tau}{\delta t} - \frac{1}{2} \right). \quad (7)$$

Usually, this surprising fact is explained by an asymptotic closeness of Boltzmann and Navier-Stokes description. However, as we will show in the following, the discrete evolution (6) can also be interpreted directly as finite difference approximation of the Navier-Stokes system.

### 3 The diffusion scaling

In dealing with LBM, it is useful to distinguish between Boltzmann and Navier-Stokes scales. In both cases, a typical distance  $L$  in the problem serves as space scale but the natural scales for velocity differ: in the Boltzmann equation, a typical speed is  $|\mathbf{c}_1|$  while in the Navier-Stokes equation, only the flow velocity  $\mathbf{u}$  is relevant, giving rise to a representative speed  $U$ . In connection with  $L$ , the velocity scales lead to the time scales  $T_{BE} = L/|\mathbf{c}_1|$  and  $T = L/U$ . A considerable difference between Boltzmann and Navier-Stokes scales now follows from the incompressibility requirement because the Boltzmann equation describes a compressible gas and incompressibility is only approximately achieved by restriction to low Mach number flows. Since the sound speed in the lattice gas is comparable to the particle speed, we conclude that  $U/|\mathbf{c}_1|$  must be small. Introducing  $\epsilon = \delta x/L$  (which should be small to obtain a reasonable spatial resolution), the considerations in [12] imply

that the Mach number must be of order  $\epsilon$  to get a consistent approximation. Since  $U$  is only a scale for the speed, we set  $U/|\mathbf{c}_1| = \epsilon$ , absorbing possible factors in the scaled velocity  $\hat{\mathbf{u}} = \mathbf{u}/U$ , and obtain  $T_{BE} = \epsilon T$ .

Due to the relation  $\delta x = |\mathbf{c}_1|\delta t$ , the scaled time step  $\Delta t_{BE} = \delta t/T_{BE}$  satisfies in relation to  $\Delta x = \delta x/L$

$$\Delta t_{BE} = \frac{\delta t}{T_{BE}} = \frac{\delta x}{L} = \Delta x \quad (8)$$

in the Boltzmann scale. However, (8) is somewhat misleading because it does not reflect the fact that many lattice Boltzmann steps are necessary to see macroscopic effects related to the flow velocity  $\mathbf{u}$ . This property is more obvious in the Navier-Stokes scales where the dimensionless time step  $\Delta t = \delta t/T$  satisfies

$$\Delta t = \frac{\delta t}{T} = \epsilon \frac{\delta t}{T_{BE}} = \epsilon \frac{\delta x}{L} = \epsilon^2 = \Delta x^2.$$

In order to write the lattice Boltzmann evolution (6) in the problem related Navier-Stokes scales, we introduce scaled quantities  $\hat{\mathbf{x}} = \mathbf{x}/L$ ,  $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{c}_1|$ ,  $\hat{t} = t/T$ ,  $\hat{\tau} = \tau/\delta t$ ,  $\hat{f}(\hat{\mathbf{x}}, \hat{\mathbf{v}}, \hat{t}) = f(L\hat{\mathbf{x}}, |\mathbf{c}_1|\hat{\mathbf{v}}, T\hat{t})$  and similar definitions for  $\hat{\rho}$  and  $\hat{\mathbf{u}} = \mathbf{u}/\epsilon$

$$\hat{f}(\hat{\mathbf{x}} + \epsilon\hat{\mathbf{v}}, \hat{\mathbf{v}}, \hat{t} + \epsilon^2) = \hat{f}(\hat{\mathbf{x}}, \hat{\mathbf{v}}, \hat{t}) - \frac{1}{\hat{\tau}} \left( \hat{f}(\hat{\mathbf{x}}, \hat{\mathbf{v}}, \hat{t}) - f^{eq}(\hat{\rho}(\hat{\mathbf{x}}, \hat{t}), \epsilon\hat{\mathbf{u}}(\hat{\mathbf{x}}, \hat{t}); \hat{\mathbf{v}}) \right)$$

(note that  $|\mathbf{c}_1| = 1$ , so that  $\mathbf{v}/|\mathbf{c}_1|$  only removes the physical unit of the velocity but does not change the length). In the following, we will always work in the Navier-Stokes scaling and therefore skip the hat-superscripts again. Also, the relations  $\Delta x = \epsilon$  and  $\Delta t = \epsilon^2$  will frequently be used. In the new notation, LBM has the form

$$f(\mathbf{x} + \mathbf{v}\Delta x, \mathbf{v}, t + \Delta t) = f(\mathbf{x}, \mathbf{v}, t) - \frac{1}{\tau} \left( f(\mathbf{x}, \mathbf{v}, t) - f^{eq}(\rho(\mathbf{x}, t), \epsilon\mathbf{u}(\mathbf{x}, t); \mathbf{v}) \right) \quad (9)$$

where  $\mathbf{x}$  is a point on a regular square lattice with spacing  $\Delta x$ . Since (9) has been obtained by a discretization of the Boltzmann equation (1), we also expect consistency to (1). However, this is only true if  $\Delta t$  and  $\Delta x$  are of the same order. Due to the relation  $\Delta t = \Delta x^2$ , we find that (9) is first order consistent in  $\Delta t$  to a modification of the Boltzmann equation which can be calculated by applying a Taylor expansion around  $\Delta x = \Delta t = 0$

$$\begin{aligned} \frac{\partial f}{\partial t} + \frac{1}{\epsilon} v_k \frac{\partial f}{\partial x_k} &= -\frac{1}{\epsilon^2 \tau} \left( f - f^{eq}(\rho, \epsilon\mathbf{u}) \right) + \frac{1}{2} v_k v_l \frac{\partial^2 f}{\partial x_k \partial x_l} \\ &+ \frac{1}{\epsilon \tau} v_k \frac{\partial}{\partial x_k} \left( f - f^{eq}(\rho, \epsilon\mathbf{u}) \right) + \frac{1}{3} \epsilon v_k v_l v_m \frac{\partial^3 f}{\partial x_k \partial x_l \partial x_m}. \end{aligned} \quad (10)$$

The space derivative of the collision operator is due to a term  $\epsilon v_k \partial^2 f / \partial t \partial x_k$  in the expansion of  $f(\mathbf{x} + \mathbf{v}\Delta x, \mathbf{v}, t + \Delta t)$ . Replacing the time part of this derivative with

the help of the modified Boltzmann equation then leads to derivatives of the collision term and to third order spatial derivatives. We remark that equation (10) is very similar to the Boltzmann equation (1) in the *diffusion scaling*  $\mathbf{x} \rightarrow \mathbf{x}/\epsilon$ ,  $t \rightarrow t/\epsilon^2$ ,  $\mathbf{u} \rightarrow \epsilon\mathbf{u}$

$$\frac{\partial f}{\partial t} + \frac{1}{\epsilon} \mathbf{v} \nabla f = -\frac{1}{\epsilon^2 \tau} \left( f - f^{eq}(\rho, \epsilon \mathbf{u}) \right). \quad (11)$$

For (11) it is well known (see [13, 14]) that, in lowest  $\epsilon$ -order, the velocity  $\mathbf{u} = \langle \mathbf{v} f \rangle / \langle f \rangle$  is a solution to the Navier-Stokes equation with  $1/Re = \tau/3$  and that the pressure is related to  $\epsilon^2$ -fluctuations of the density  $\rho = \langle f \rangle$ . Applying the same analysis to (10), a similar result is obtained (only the Reynolds number changes to

$$\frac{1}{Re} = \frac{1}{3} \left( \tau - \frac{1}{2} \right) \quad (12)$$

because of the second order  $f$  derivatives). Since (10) is, up to terms of order  $\Delta t$ , just the lattice Boltzmann evolution (6) written in the Navier-Stokes scales (which are natural scales for the macroscopic problem), we conclude that LBM really approximates the incompressible Navier-Stokes equation. In the following section, we discuss the limit  $\epsilon \rightarrow 0$  for a system of velocity moments which is equivalent to the velocity discrete Boltzmann equation.

## 4 Equivalent moment systems

Generally speaking, equation (11) consists of a linear hyperbolic differential operator and a nonlinear, stiff relaxation term on the right hand side. Obviously, this behavior does not change under a linear transformation of variables. In particular, if we choose an invertible linear mapping which includes the components

$$f \mapsto \begin{pmatrix} \langle f \rangle \\ \langle \mathbf{v}/\epsilon f \rangle \end{pmatrix} = \begin{pmatrix} \rho \\ \rho \mathbf{u} \end{pmatrix} \quad (13)$$

the resulting system contains mass and momentum equations as a subsystem (see also [15] and [16] for a similar reformulation of LBM). The new system will again be hyperbolic with stiff relaxation terms which suggests the interpretation as *relaxation system*. To make these ideas more precise, we extend (13) to an invertible mapping by considering additional velocity moments. Apart from

$$Q_0(\mathbf{v}) = 1, \quad Q_1(\mathbf{v}) = v_1/\epsilon, \quad Q_2(\mathbf{v}) = v_2/\epsilon$$

we take

$$Q_3(\mathbf{v}) = \frac{1}{\epsilon^2} \left( v_1^2 - \frac{1}{3} \right), \quad Q_4(\mathbf{v}) = \frac{1}{\epsilon^2} v_1 v_2, \quad Q_5(\mathbf{v}) = \frac{1}{\epsilon^2} \left( v_2^2 - \frac{1}{3} \right)$$

and

$$Q_6(\mathbf{v}) = \frac{(3|\mathbf{v}|^2 - 4)v_1}{\epsilon^3} \quad Q_7(\mathbf{v}) = \frac{(3|\mathbf{v}|^2 - 4)v_2}{\epsilon^3} \quad Q_8(\mathbf{v}) = \frac{9|\mathbf{v}|^4 - 15|\mathbf{v}|^2 + 2}{\epsilon^4}$$

The polynomials  $Q_i$  are chosen in such a way that they are mutually  $f^*$  orthogonal, i.e.  $\langle Q_i Q_j f^* \rangle = 0$  for  $i \neq j$ . Since also  $\langle Q_i^2 f^* \rangle > 0$  for all  $i$ , one can show that

$$\mathbf{Q}f = (\langle Q_0 f \rangle, \dots, \langle Q_8 f \rangle)^T$$

is a linear, invertible mapping. Indeed, given  $\mathbf{M} \in \mathbb{R}^9$ , we have

$$\mathbf{Q}^{-1}\mathbf{M} = \sum_{i=0}^8 \frac{M_i Q_i}{\langle Q_i^2 f^* \rangle} f^*, \quad \text{or} \quad f = \sum_{i=0}^8 \frac{\langle Q_i f \rangle}{\langle Q_i^2 f^* \rangle} Q_i f^*. \quad (14)$$

Taking into account that

$$\begin{aligned} \langle Q_0^2 f^* \rangle^{-1} &= 1, & \langle Q_1^2 f^* \rangle^{-1} &= 3\epsilon^2, & \langle Q_2^2 f^* \rangle^{-1} &= 3\epsilon^2, \\ \langle Q_3^2 f^* \rangle^{-1} &= 9\epsilon^4/2, & \langle Q_4^2 f^* \rangle^{-1} &= 9\epsilon^4, & \langle Q_5^2 f^* \rangle^{-1} &= 9\epsilon^4/2 \end{aligned}$$

we see that  $f^{eq}(\rho, \epsilon \mathbf{u}; \mathbf{v})$  in (4) is precisely of the form (14) with

$$\mathbf{M}^{eq} = \mathbf{Q}f^{eq}(\rho, \epsilon \mathbf{u}) = (\rho, \rho u_1, \rho u_2, \rho u_1^2, \rho u_1 u_2, \rho u_2^2, 0, 0, 0)^T. \quad (15)$$

By applying  $\mathbf{Q}$  to (11), a system of equations for the moments  $\mathbf{M} = \mathbf{Q}f$  is obtained

$$\frac{\partial \mathbf{M}}{\partial t} + \frac{1}{\epsilon} \mathbf{Q} v_k \mathbf{Q}^{-1} \frac{\partial \mathbf{M}}{\partial x_k} = -\frac{1}{\epsilon^2 \tau} (\mathbf{M} - \mathbf{M}^{eq}) \quad (16)$$

(where  $v_k$  is used as abbreviation for the product operator  $f(\mathbf{v}) \mapsto v_k f(\mathbf{v})$ ). To get (16) in a more convenient form, we introduce auxiliary names for the variables. Apart from  $\langle Q_0 f \rangle = \rho$  and  $\langle Q_i f \rangle = \rho u_i$  for  $i = 1, 2$ , the second order moments form a symmetric tensor  $\Theta$ , the third order polynomials lead to a vector  $\mathbf{q}$  and  $Q_8$  gives rise to a scalar quantity  $s$

$$\Theta = \begin{pmatrix} \langle Q_3 f \rangle & \langle Q_4 f \rangle \\ \langle Q_4 f \rangle & \langle Q_5 f \rangle \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} \langle Q_6 f \rangle \\ \langle Q_7 f \rangle \end{pmatrix}, \quad s = \langle Q_8 f \rangle.$$

In these variables, the first two equations of (16) are related to mass and momentum conservation

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \text{div } \rho \mathbf{u} &= 0 \\ \frac{\partial \rho \mathbf{u}}{\partial t} + \text{div } \Theta + \frac{1}{3\epsilon^2} \nabla \rho &= 0 \end{aligned} \quad (17)$$

Here, the divergence is applied to the rows of  $\Theta$ . The equation for  $\Theta$  is

$$\frac{\partial \Theta}{\partial t} + \frac{1}{3} \begin{pmatrix} \partial_{x_2} q_2 & \partial_{x_2} q_1 + \partial_{x_1} q_2 \\ \partial_{x_2} q_1 + \partial_{x_1} q_2 & \partial_{x_1} q_1 \end{pmatrix} = -\frac{1}{\epsilon^2 \tau} (\Theta - \rho \mathbf{u} \otimes \mathbf{u} + \frac{2\tau}{3} S[\rho \mathbf{u}]) \quad (18)$$



where

$$S_{ij}[\rho\mathbf{u}] = \frac{1}{2} \left( \frac{\partial \rho u_i}{\partial x_j} + \frac{\partial \rho u_j}{\partial x_i} \right).$$

Finally, the third and fourth order moments satisfy

$$\begin{aligned} \frac{\partial \mathbf{q}}{\partial t} + \frac{1}{6} \nabla s &= -\frac{1}{\epsilon^2 \tau} \left( \mathbf{q} + \tau \operatorname{div} \begin{pmatrix} \Theta_{22} & 2\Theta_{12} \\ 2\Theta_{12} & \Theta_{11} \end{pmatrix} \right) \\ \frac{\partial s}{\partial t} &= -\frac{1}{\epsilon^2 \tau} (s + 4\tau \operatorname{div} \mathbf{q}) \end{aligned} \quad (19)$$

The diffusion limit  $\epsilon \rightarrow 0$  of the system (17), (18) and (19) is easily determined. First we assume that  $\rho$  is of the form

$$\rho = \bar{\rho}(1 + 3\epsilon^2 p) \quad (20)$$

with a constant  $\bar{\rho} > 0$  and an order one function  $p$ . Relation (20) is motivated by (17). Indeed, since all terms are scaled to order one,  $\nabla \rho / \epsilon^2$  can only be balanced if  $\nabla \rho = \mathcal{O}(\epsilon^2)$ . Using (20), equation (17) transforms into

$$\begin{aligned} \frac{\partial p}{\partial t} + \frac{1}{3\epsilon^2} \operatorname{div} \mathbf{u} &= -\operatorname{div} p \mathbf{u} \\ \frac{\partial \mathbf{u}}{\partial t} + \operatorname{div} \frac{1}{\bar{\rho}} \Theta + \nabla p &= -3\epsilon^2 \frac{\partial p \mathbf{u}}{\partial t} \end{aligned} \quad (21)$$

and for  $\epsilon \rightarrow 0$ , equation (18) yields in lowest order

$$\frac{1}{\bar{\rho}} \Theta = \mathbf{u} \otimes \mathbf{u} - \frac{2\tau}{3} S[\mathbf{u}]. \quad (22)$$

Since (19) decouples completely from the other equations (in lowest order) and since  $\operatorname{div} S[\mathbf{u}] = (\Delta + \nabla \operatorname{div}) \mathbf{u} / 2$ , we obtain from (21) and (22) as limiting system the incompressible Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + \operatorname{div} \mathbf{u} \otimes \mathbf{u} + \nabla p = \frac{\tau}{3} \Delta \mathbf{u}, \quad \operatorname{div} \mathbf{u} = 0 \quad (23)$$

The Reynolds number is related to the relaxation time by  $1/Re = \tau/3$ . We remark that (17), (18), and (19) is a *relaxation system* for (23) as described for nonlinear hyperbolic systems in [17]. Moreover, the equations for  $\mathbf{q}$  and  $s$  are not relevant in lowest  $\epsilon$ -order. This indicates that the original discrete velocity equation carries too much information if the only aim is to approximate the limiting Navier-Stokes system. In fact, one can set up a lattice Boltzmann method based on only six discrete velocities which is compatible with a regular hexagonal space grid [18]. In this case, the equivalent moment system just consists of equations for mass, momentum and  $\Theta$  which obviously is the minimal requirement in view of the underlying relaxation system. Thus, the overhead of three variables in the nine-velocity model is only due to the interplay between the required symmetries of the velocity space  $\mathcal{V}$  and the condition that the grid should be a square lattice.

The basic idea of relaxation schemes, to replace nonlinear conservation systems by linear equations with nonlinear relaxation terms, comes with the price of introducing new variables. If the extended system is solved in a bounded domain, this leads to the problem of prescribing boundary conditions for the extra variables. In the system (17), (18), and (19), some of the variables still have a clear physical meaning (for example,  $\Theta$  is the momentum flux up to the pressure term). This kind of interpretation is lost for higher moments (like  $s$ ) so that it is difficult to find suitable boundary conditions for particular physical situations. A similar problem appears in the classical theory of moment systems derived from the Boltzmann equation and is yet unsolved [19, 20]. Translated back into the original kinetic formulation, the problem to determine boundary values for higher moments is recovered in the sense that now particle densities have to be prescribed for all directions which enter the domain at a boundary.

As we have seen in the previous section, the discrete lattice Boltzmann evolution is consistent to (10) which is a modification of the original Boltzmann equation. Repeating the above argument for (10) instead of (11), we just have to consider additional contributions due to

$$\mathbf{Q} \frac{1}{\epsilon \tau} v_k \frac{\partial}{\partial x_k} \left( f - f^{eq}(\rho, \epsilon \mathbf{u}) \right) = \frac{1}{\epsilon} \mathbf{Q} v_k \mathbf{Q}^{-1} \frac{\partial}{\partial x_k} \frac{1}{\tau} (\mathbf{M} - \mathbf{M}^{eq}),$$

adding to the first order derivatives in (16), as well as

$$\frac{1}{2} \frac{\partial^2}{\partial x_k \partial x_l} \mathbf{Q} v_k v_l \mathbf{Q}^{-1} \mathbf{M} + \frac{\epsilon}{3} \frac{\partial^3}{\partial x_k \partial x_l \partial x_m} \mathbf{Q} v_k v_l v_m \mathbf{Q}^{-1} \mathbf{M}$$

which adds second and third order derivative terms to the moment system. Introducing  $\omega = 1/\tau$  and  $\Theta^\omega = (1 - \omega)\Theta + \omega \rho \mathbf{u} \otimes \mathbf{u}$ , mass and momentum equation (17) now have the form

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{u} &= \frac{1}{6} \Delta \rho \\ \frac{\partial \rho \mathbf{u}}{\partial t} + \operatorname{div} \Theta^\omega + \frac{1}{3\epsilon^2} \nabla \rho &= \frac{1}{6} (\Delta + 2\nabla \operatorname{div}) \rho \mathbf{u} - \frac{1}{18} \nabla \Delta \rho. \end{aligned} \quad (24)$$

Equation (18) is modified to

$$\begin{aligned} \frac{\partial \Theta}{\partial t} + \frac{1 - \omega}{3} \begin{pmatrix} \partial_{x_2} q_2 & \partial_{x_2} q_1 + \partial_{x_1} q_2 \\ \partial_{x_2} q_1 + \partial_{x_1} q_2 & \partial_{x_1} q_1 \end{pmatrix} \\ = -\frac{\omega}{\epsilon^2} (\Theta - \rho \mathbf{u} \otimes \mathbf{u}) - \frac{1}{3\epsilon^2} \left( 2S[\rho \mathbf{u}] - \frac{1}{9} \nabla \otimes \nabla \rho \right) + B \end{aligned} \quad (25)$$

where  $(\nabla \otimes \nabla \rho)_{ij} = \partial_{x_i} \partial_{x_j} \rho$  and  $B$  contains third derivatives of momentum and second derivatives of  $\Theta$  and  $\rho \mathbf{u} \otimes \mathbf{u}$ . Since  $B$  is irrelevant in the lowest  $\epsilon$ -order, we omit details. Similarly, the equations for  $\mathbf{q}$  and  $s$  given in (19) are extended by second and third order derivatives but, in the limit  $\epsilon \rightarrow 0$ , they have no influence

on the equations for  $\rho$  and  $\mathbf{u}$ . Inserting the assumption (20) into the equation for  $\rho$  again leads to  $\text{div } \mathbf{u} = \mathcal{O}(\epsilon^2)$  and (24) turns into

$$\frac{\partial \mathbf{u}}{\partial t} + \text{div} \left( \frac{(1-\omega)}{\bar{\rho}} \Theta + \omega \mathbf{u} \otimes \mathbf{u} \right) + \nabla p = \frac{1}{6} \Delta \mathbf{u} + \mathcal{O}(\epsilon^2), \quad \text{div } \mathbf{u} = \mathcal{O}(\epsilon^2)$$

Moreover, from (25) we see that

$$\frac{1}{\bar{\rho}} \Theta = \mathbf{u} \otimes \mathbf{u} - \frac{2\tau}{3} S[\mathbf{u}] + \mathcal{O}(\epsilon^2) \quad (26)$$

so that with

$$\frac{2\tau(1-\omega)}{3} \text{div} S[\mathbf{u}] = \frac{\tau-1}{3} (\Delta \mathbf{u} + \nabla \text{div } \mathbf{u}) = \frac{\tau-1}{3} \Delta \mathbf{u} + \mathcal{O}(\epsilon^2), \quad (27)$$

the velocity  $\mathbf{u}$  in (24) satisfies

$$\frac{\partial \mathbf{u}}{\partial t} + \text{div } \mathbf{u} \otimes \mathbf{u} + \nabla p = \frac{1}{3} \left( \tau - \frac{1}{2} \right) \Delta \mathbf{u} + \mathcal{O}(\Delta t), \quad \text{div } \mathbf{u} = \mathcal{O}(\Delta t).$$

From this result we can draw three conclusions. First, since the discrete lattice Boltzmann evolution is consistent to (10), it is also consistent to the equivalent moment system including (24), (25) and thus to the incompressible Navier-Stokes equation. Secondly, the equations (24), (25) form a relaxation-type system for the incompressible Navier-Stokes equation. Note, however, that the nonlinearity in the convective part of the momentum equation is not completely replaced by a new variable. Instead, we find the linear combination  $\text{div}((1-\omega)\Theta + \omega\rho\mathbf{u}\otimes\mathbf{u})$  so that the equations are a combination of a direct and a relaxation system. Another difference to usual relaxation systems is the coupling between the relaxation parameter  $\tau\epsilon^2$  and the space and time steps  $\Delta x = \epsilon$ ,  $\Delta t = \epsilon^2$ . Finally, if LBM is considered only as numerical method to approximate the Navier-Stokes equation, the information contained in the highest order moments  $\mathbf{q}$ ,  $s$  is not needed. If we rewrite  $f$  in terms of its moments using the unique representation (14)

$$f = \left( \rho + 3\epsilon\rho u_k v_k + \frac{9\epsilon^2}{2} \Theta_{kl} (v_k v_l - \delta_{kl}/3) + \frac{3\epsilon^3}{2} (3|\mathbf{v}|^2 - 4)v_k q_k + \frac{\epsilon^4}{16} P_8 s \right) f^* \quad (28)$$

(with  $P_8 = \epsilon^4 Q_8$ ) we can use the fact that  $\mathbf{q}$  and  $s$  are not required in the limit and that the only relevant information in  $\Theta$  is relation (26). Setting  $\mathbf{q} = \mathbf{0}$  and  $s = 0$  in (28) and replacing  $\Theta$  with  $\rho(\mathbf{u} \otimes \mathbf{u} - \frac{2\tau}{3} S[\mathbf{u}])$ , as suggested by (26), we obtain the distribution function

$$F(\rho, \mathbf{u}; \mathbf{v}) = \rho \left( 1 + 3u_k v_k + \frac{9}{2} \left( u_k u_l - \frac{2\tau}{3} S_{kl}[\mathbf{u}] \right) (v_k v_l - \delta_{kl}/3) \right) f^*(\mathbf{v}).$$

where  $S[\mathbf{u}]$  has to be calculated by taking derivatives of  $\mathbf{u}$ . In [21], a distribution function similar to  $F$  has been used to set up suitable boundary and initial values

for the lattice Boltzmann method. Since the higher moments  $\mathbf{q}$  and  $s$  are negligible and  $\Theta$  is given essentially through  $\rho$  and  $\mathbf{u}$ ,  $F$  can be viewed as an approximate inverse of the moment map  $f \mapsto (\langle f \rangle, \langle \mathbf{v}f \rangle / \epsilon)$ . This observation can be used to translate initial and boundary conditions which are given in terms of  $\rho$  and  $\mathbf{u}$  into corresponding conditions for the particle densities (see [21] for details). In [22], a scheme has been introduced which is similar to the lattice Boltzmann method but which is based on  $F$  and avoids storing all particle densities.

While the moment system is obtained by applying  $\mathbf{Q}$  to the continuous version (10) of the lattice Boltzmann evolution (9), a discretization of the moment system follows by applying  $\mathbf{Q}$  directly to (9). In fact, since  $\mathbf{Q}$  is an invertible mapping, this discretization of the moment system is equivalent to LBM. To simplify notation, we first rewrite (9) as in [8] by using  $\omega = 1/\tau$ , changing  $\mathbf{x}$  to  $\mathbf{x} - \mathbf{v}\Delta x$  and denoting the number of the time step by an upper index

$$f^{n+1}(\mathbf{x}, \mathbf{v}) = (1 - \omega)f^n(\mathbf{x} - \mathbf{v}\Delta x, \mathbf{v}) + \omega f^{eq}(\rho^n(\mathbf{x} - \mathbf{v}\Delta x), \epsilon \mathbf{u}^n(\mathbf{x} - \mathbf{v}\Delta x); \mathbf{v}). \quad (29)$$

Applying  $\mathbf{Q}$  to (29) amounts to multiplying the equation by  $Q_i$  and integrating over  $\mathbf{v}$ . On the left hand side, we obtain immediately the  $i$ -th component of the moment vector at the new time step

$$M_i^{n+1}(\mathbf{x}) = \langle Q_i f^{n+1}(\mathbf{x}, \mathbf{v}) \rangle.$$

To treat the terms on the right hand side, we use the unique representation (14) of distribution functions in terms of their moments

$$\begin{aligned} (1 - \omega)f^n(\mathbf{x} - \mathbf{v}\Delta x, \mathbf{v}) + \omega f^{eq}(\rho^n(\mathbf{x} - \mathbf{v}\Delta x), \epsilon \mathbf{u}^n(\mathbf{x} - \mathbf{v}\Delta x); \mathbf{v}) \\ = \sum_{j=0}^8 \frac{Q_j(\mathbf{v})}{\langle Q_j^2 f^* \rangle} \left( (1 - \omega)M_j^n + \omega M_j^{n,eq} \right) (\mathbf{x} - \mathbf{v}\Delta x) f^*(\mathbf{v}) \end{aligned}$$

where we have set  $\mathbf{M}^n = \mathbf{Q}f^n$  and  $\mathbf{M}^{n,eq} = \mathbf{Q}f^{eq}(\rho^n, \epsilon \mathbf{u}^n)$ . Altogether, we obtain the equivalent form of (29)

$$M_i^{n+1}(\mathbf{x}) = \sum_{j=0}^8 \frac{1}{\langle Q_j^2 f^* \rangle} \langle Q_i Q_j \left( (1 - \omega)M_j^n + \omega M_j^{n,eq} \right) (\mathbf{x} - \mathbf{v}\Delta x) f^* \rangle \quad (30)$$

with  $i = 0, \dots, 8$ . Again,  $\mathbf{x}$  is restricted to a regular square lattice with spacing  $\Delta x$ . In the next section, we show that (30) is, in fact, a compact way of writing a finite difference discretization.

## 5 Microscopic transport and finite difference stencils

In order to interpret the discretization (30), we analyze expressions of the form

$$\langle \psi(\mathbf{x} - \mathbf{v}\Delta x) P f^* \rangle \quad (31)$$

where  $\psi$  is a smooth function,  $f^*$  is defined in (5), and  $P$  is a function in  $\mathbf{v}$  (typically a polynomial). Note that  $g(\mathbf{x}, \mathbf{v}, t) = \psi(\mathbf{x} - t\mathbf{v}/\epsilon)f^*(\mathbf{v})$  is the exact solution of the transport problem

$$\frac{\partial g}{\partial t} + \frac{1}{\epsilon}\mathbf{v} \cdot \nabla g = 0, \quad g(\mathbf{x}, \mathbf{v}, 0) = \psi(\mathbf{x})f^*(\mathbf{v}).$$

Hence, (31) is the  $P$ -moment of the solution of the transport problem at time  $\Delta t = \epsilon^2$  (taking into account that  $\Delta x = \epsilon$ ).

Introducing the notation  $\mathbf{x}_{ij} = (i\Delta x, j\Delta x)$  and using the definition of  $\langle \cdot \rangle$ , we find

$$\langle \psi(\mathbf{x}_{ij} - \mathbf{v}\Delta x)(Pf^*)(\mathbf{v}) \rangle = \sum_{m=0}^8 \psi(\mathbf{x}_{ij} + \mathbf{c}_m\Delta x)(Pf^*)(-\mathbf{c}_m) = \sum_{k,l=-1}^1 \alpha_{kl}\psi(\mathbf{x}_{i+k,j+l})$$

so that (31) is, in fact, a finite difference expression. Usually, the coefficients  $\alpha_{kl}$  are arranged in a table (the so called stencil). Here, we have

$$\begin{bmatrix} \alpha_{-11} & \alpha_{01} & \alpha_{11} \\ \alpha_{-10} & \alpha_{00} & \alpha_{10} \\ \alpha_{-1-1} & \alpha_{0-1} & \alpha_{1-1} \end{bmatrix} = \begin{bmatrix} Pf^*(\mathbf{c}_8) & Pf^*(\mathbf{c}_4) & Pf^*(\mathbf{c}_7) \\ Pf^*(\mathbf{c}_1) & Pf^*(\mathbf{c}_0) & Pf^*(\mathbf{c}_3) \\ Pf^*(\mathbf{c}_5) & Pf^*(\mathbf{c}_2) & Pf^*(\mathbf{c}_6) \end{bmatrix} \quad (32)$$

If  $f^*$  is kept fixed, we can thus identify polynomials with finite difference approximations.

To find out the approximation properties of the stencil belonging to a particular polynomial, we replace  $\psi(\mathbf{x} - \mathbf{v}\Delta x)$  by a Taylor expansion

$$\begin{aligned} \langle \psi(\mathbf{x} - \mathbf{v}\Delta x)Pf^* \rangle &= \langle Pf^* \rangle \psi(\mathbf{x}) - \Delta x \langle v_i Pf^* \rangle \partial_{x_i} \psi(\mathbf{x}) \\ &+ \frac{\Delta x^2}{2} \langle v_i v_j Pf^* \rangle \partial_{x_i} \partial_{x_j} \psi(\mathbf{x}) - \frac{\Delta x^3}{6} \langle v_i v_j v_k Pf^* \rangle \partial_{x_i} \partial_{x_j} \partial_{x_k} \psi(\mathbf{x}) + \mathcal{O}(\Delta x^4). \end{aligned}$$

Note that the coefficients in front of the derivatives are just velocity moments of  $f^*$ . An immediate consequence of the symmetry

$$f^*(\mathbf{v}) = f^*(-\mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{V}$$

is that all odd derivatives in the Taylor expansion vanish if  $P(-\mathbf{v}) = P(\mathbf{v})$  and, vice versa, all even derivatives disappear provided  $P(-\mathbf{v}) = -P(\mathbf{v})$ . Hence, polynomials  $P$  involving only even (odd) orders are related to second order approximations of even (odd) derivatives. These observations apply in particular to the discretization (30) because all polynomials  $Q_i$  are either even or odd.

The following examples are given in the form

$$\text{polynomial} \leftrightarrow \text{stencil} \leftrightarrow \text{approximated operator.}$$

We begin with  $P = Q_0Q_0/\langle Q_0^2f^* \rangle = 1$  which essentially describes the nine-point formula for the Laplacian

$$1 \leftrightarrow \frac{1}{36} \begin{bmatrix} 1 & 4 & 1 \\ 4 & 16 & 4 \\ 1 & 4 & 1 \end{bmatrix} \leftrightarrow 1 + \frac{\Delta x^2}{6} \Delta + \mathcal{O}(\Delta x^4). \quad (33)$$

Our next example is based on  $P = Q_0Q_i/\langle Q_i^2f^* \rangle = 3\epsilon v_i$  for  $i = 1, 2$  which gives rise to first derivatives

$$-\frac{3v_1}{\Delta x} \leftrightarrow \frac{1}{12\Delta x} \begin{bmatrix} -1 & 0 & 1 \\ -4 & 0 & 4 \\ -1 & 0 & 1 \end{bmatrix} \leftrightarrow \frac{\partial}{\partial x_1} + \mathcal{O}(\Delta x^2) \quad (34)$$

and

$$-\frac{3v_2}{\Delta x} \leftrightarrow \frac{1}{12\Delta x} \begin{bmatrix} 1 & 4 & 1 \\ 0 & 4 & 0 \\ -1 & -4 & -1 \end{bmatrix} \leftrightarrow \frac{\partial}{\partial x_2} + \mathcal{O}(\Delta x^2). \quad (35)$$

We remark that both stencils can be viewed as convex combinations of standard central difference approximations. For example, the  $\partial_{x_1}$ -stencil in (34) can be written as

$$\frac{1}{12} \begin{bmatrix} -1 & 0 & 1 \\ -4 & 0 & 4 \\ -1 & 0 & 1 \end{bmatrix} = \frac{1}{6} \left( \frac{1}{2} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{4}{2} \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \right)$$

This observation applies to many stencils found in (30). However, the convex combinations are not always those of (34) and (35). Related to the odd polynomial  $P = Q_2Q_4/\langle Q_4^2f^* \rangle = 9\epsilon v_1v_2^2$ , we get

$$-\frac{9v_1v_2^2}{\Delta x} \leftrightarrow \frac{1}{4\Delta x} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \leftrightarrow \frac{\partial}{\partial x_1} + \mathcal{O}(\Delta x^2). \quad (36)$$

and similarly for  $P = Q_1Q_4/\langle Q_4^2f^* \rangle = 9\epsilon v_1^2v_2$

$$-\frac{9v_1^2v_2}{\Delta x} \leftrightarrow \frac{1}{4\Delta x} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix} \leftrightarrow \frac{\partial}{\partial x_2} + \mathcal{O}(\Delta x^2). \quad (37)$$

The third approximation of first derivatives is obtained, for example, from the combination  $P = Q_5Q_6/\langle Q_6^2f^* \rangle = 3/2\epsilon v_1(v_2^2 - 1/3)(3v_1^2 + 3v_2^2 - 4)$

$$-\frac{3}{2\Delta x} (3v_1^2 - 1)(3v_1^2 + 3v_2^2 - 4) \leftrightarrow \frac{1}{6\Delta x} \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \leftrightarrow \frac{\partial}{\partial x_1} + \mathcal{O}(\Delta x^2). \quad (38)$$

The corresponding approximation of  $\partial_{x_2}$  follows by exchanging the roles of the coordinates. For the stencil, this exchange amounts to a reflection of the weights at the diagonal and, in view of (32), leads to the transformation  $P(v_1, v_2) \rightarrow P(v_2, v_1)$  of the polynomial. Note that all polynomials  $Q_i$  respect this symmetry, i.e.  $(v_1, v_2) \mapsto Q(v_2, v_1)$  is again some  $Q_j$ .

Continuing with the even polynomials

$$\begin{pmatrix} Q_0Q_3/\langle Q_3^2f^* \rangle & Q_0Q_4/\langle Q_4^2f^* \rangle \\ Q_0Q_4/\langle Q_4^2f^* \rangle & Q_0Q_5/\langle Q_5^2f^* \rangle \end{pmatrix} = \frac{9}{2}\epsilon^2 \begin{pmatrix} v_1^2 - 1/3 & 2v_1v_2 \\ 2v_1v_2 & v_2^2 - 1/3 \end{pmatrix}$$

we find the second order derivative

$$\frac{9}{\Delta x^2} \left( v_1^2 - \frac{1}{3} \right) \leftrightarrow \frac{1}{6\Delta x^2} \begin{bmatrix} 1 & -2 & 1 \\ 4 & -8 & 4 \\ 1 & -2 & 1 \end{bmatrix} \leftrightarrow \frac{\partial^2}{\partial x_1^2} + \mathcal{O}(\Delta x^2) \quad (39)$$

and the reflected version for  $9(v_2^2 - 1/3)$ . A mixed derivative is given by

$$\frac{9v_1v_2}{\Delta x^2} \leftrightarrow \frac{1}{4\Delta x^2} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \leftrightarrow \frac{\partial^2}{\partial x_1 \partial x_2} + \mathcal{O}(\Delta x^2). \quad (40)$$

We remark that the size of the stencils is directly related to the number of discrete velocities in the lattice Boltzmann model. In particular, usual central differences and the five point stencil for the Laplacian are obtained with a model consisting only of  $\mathbf{c}_0, \dots, \mathbf{c}_4$ . Standard lattice Boltzmann methods with this particular choice of discrete velocities, however, are not consistent to the Navier-Stokes equation (see [23]).

## 6 LBM as finite difference scheme

For the special case  $\tau = 1$ , the lattice Boltzmann evolution (9) simplifies to

$$f(\mathbf{x} + \mathbf{v}\Delta x, t + \Delta t) = f^{eq}(\rho(\mathbf{x}, t), \epsilon \mathbf{u}(\mathbf{x}, t); \mathbf{v}). \quad (41)$$

Since  $\omega = 1/\tau = 1$ , the equivalent moment system (30) has the form

$$M_i^{n+1}(\mathbf{x}) = \sum_{j=0}^8 \frac{1}{\langle Q_j^2 f^* \rangle} \langle Q_i Q_j M_j^{n,eq}(\mathbf{x} - \mathbf{v}\Delta x) f^* \rangle. \quad (42)$$

Note that  $f^{eq}$ , and thus also  $\mathbf{M}^{eq}$ , depend only on  $\rho$  and  $\mathbf{u}$  so that the lattice Boltzmann evolution (with  $\tau = 1$ ) is formulated completely in terms of the flow variables. In particular, (42) only has to be considered for  $i = 0, 1, 2$  and since  $M_i^{eq} = 0$  for  $i \geq 6$  (see (15)) we can rewrite (42) as

$$\begin{aligned} \rho^{n+1}(\mathbf{x}) &= \sum_{j=0}^5 \frac{1}{\langle Q_j^2 f^* \rangle} \langle Q_0 Q_j M_j^{n,eq}(\mathbf{x} - \mathbf{v}\Delta x) f^* \rangle \\ (\rho \mathbf{u})^{n+1}(\mathbf{x}) &= \sum_{j=0}^5 \frac{1}{\langle Q_j^2 f^* \rangle} \langle \left( \begin{matrix} Q_1 \\ Q_2 \end{matrix} \right) Q_j M_j^{n,eq}(\mathbf{x} - \mathbf{v}\Delta x) f^* \rangle \end{aligned} \quad (43)$$

In [24] it has been shown that (43) falls into the class of *kinetic schemes* which are routinely used in CFD (see [25]). The particular version (43) is a finite difference

realization of the general concept of kinetic schemes. Indeed, using the identification of the polynomials  $Q_0 Q_j / \langle Q_j^2 f^* \rangle$  with finite difference stencils as presented in Section 5, we can rewrite the equation for  $\rho^{n+1}$  and get

$$\rho^{n+1} = \rho^n - \epsilon \Delta x D_k \rho^n u_k^n + \frac{\Delta x^2}{6} L \rho^n + \frac{\epsilon^2 \Delta x^2}{2} D_{kl} \rho u_k^n u_l^n \quad (44)$$

where  $D_1, D_2$  are the stencils (34) and (35),  $L$  is the discretization of the Laplacian based on (33), and  $D_{kl}$  are the second order derivatives given by (39) and (40). Similarly, we can write the momentum equation in (43) as

$$\begin{aligned} (\rho u_1)^{n+1} = \rho^n u_1^n - \epsilon \Delta x (D_1 \rho^n u_1^n u_1^n + \tilde{D}_2 \rho^n u_1^n u_2^n) - \frac{\Delta x}{3\epsilon} D_1 \rho^n \\ + \frac{\Delta x^2}{6} (L \rho^n u_1^n + 2 D_{1k} \rho^n u_k^n) \end{aligned} \quad (45)$$

where  $\tilde{D}_2$  is the stencil (37) (for the second component of momentum,  $\tilde{D}_1$  given by (36) and  $D_2$  are used correspondingly). Since  $\epsilon \Delta x = \epsilon^2 = \Delta t$ , we see that (44) and (45) are indeed approximations of (24). (The discretization of  $\nabla \Delta \rho / 18$  in the momentum equation (24) is contained in  $\frac{\Delta x}{3\epsilon} D_k \rho$  which has a leading factor  $1/\epsilon^2$  after division by  $\Delta t$ . Hence, the third order parts in the discretizations  $D_k$  appear in relevant order.) While (45) essentially approximates the Navier-Stokes equation, (44) assures the approximate divergence-free condition. Basically, it is a pseudo-compressibility method [26, 27] which can be seen by setting  $\rho^n = \bar{\rho}(1 + 3\epsilon^2 p^n)$  in (44) which leads to

$$\frac{p^{n+1} - p^n}{\Delta t} + \frac{1}{3\epsilon^2} D_i u_i^n + D_i (p^n u_i^n) = \frac{1}{6} L p^n + \frac{\epsilon^2}{2\bar{\rho}} D_{kl} \rho u_k^n u_l^n.$$

This equation for the pressure contains elements of Chorin's artificial compressibility method [28] to replace  $\text{div } \mathbf{u} = 0$  by the equation

$$\frac{\partial p}{\partial t} + \frac{1}{\epsilon^2} \text{div } \mathbf{u} = 0$$

and of the pressure stabilization method

$$\frac{1}{\epsilon^2} \text{div } \mathbf{u} - \Delta p = 0$$

which was originally used in [29]. However, the convection term  $\text{div } p \mathbf{u}$  as well as the higher order nonlinear terms  $\epsilon^2 \partial_{x_i} \partial_{x_j} \rho u_i u_j$ , which appear in the discretization, are usually not considered.

Thus, in the case  $\omega = 1$ , the lattice Boltzmann method is nothing but a direct finite difference discretization of the Navier-Stokes equation (based on second order accurate nine-point stencils) together with a pseudo-compressibility approach.



As far as the structure of the stencils is concerned, it is interesting to note that first order derivatives are discretized differently (by  $D_k$  or  $\tilde{D}_k$ ), depending on the term they appear in (similarly, the discretization of the Laplacian  $L$  differs from the combination  $D_{11} + D_{22}$  of the other second order derivatives). We also remark that the original lattice Boltzmann evolution (41) together with the averaging  $\langle f^{n+1} \rangle$  and  $\langle \mathbf{v}/\epsilon f^{n+1} \rangle$  is an efficient way to evaluate the relatively large stencils. The evaluation mechanism can be described as follows: first, information about the old data  $\rho^n$  and  $\mathbf{u}^n$  is preprocessed at each node by multiplication with weights and summation. Then, each node receives preprocessed data from its neighbors (eight values are communicated per node) and calculates new quantities for  $\rho$  and  $\mathbf{u}$  by averaging. Compared to that, an equivalent, direct implementation of the stencils needs more communication. Indeed, just to calculate  $\nabla\rho$  with the stencils (34) and (35),  $\rho$  has to be obtained from all neighbors which already amounts to a communication of eight values. On top of that,  $u_1$  and  $u_2$  are needed from the neighboring sites which increases communication by a factor three. Even if standard central differences are used to discretize the Navier-Stokes equation, at least three values from four neighboring sites have to be exchanged which amounts to a higher load of communication compared to LBM (despite the fact that LBM is based on larger stencils). Thus, LBM takes advantage of structural properties of the Navier-Stokes equation (i.e. the connection to the Boltzmann equation) which are neglected by standard schemes. In fact, LBM benefits two-fold: first, the exchange of data among grid points is reduced because the discretization of a scalar equation obviously requires less communication than the discretization of a system of equations. Secondly, a simple (and therefore fast) splitting scheme can be chosen as discretization of the Boltzmann equation because discretization errors have the correct structure to be incorporated in the viscous terms of the Navier-Stokes equation. Of course, these advantages can easily turn into disadvantages if the lattice Boltzmann approach should be applied to modifications of the Navier-Stokes equation for which no natural kinetic counterpart is available. (In such cases, finding a suitable Boltzmann equation which is related to the modified system can be very difficult or even impossible.)

Before we go over to the case  $\omega \neq 1$ , let us mention an auxiliary result which we need later: if (41) is replaced by

$$f(\mathbf{x} + \mathbf{v}(k\Delta x), \mathbf{v}, t + k\Delta t) = f^{eq}(\rho(\mathbf{x}, t), \epsilon\mathbf{u}(\mathbf{x}, t); \mathbf{v})$$

with some fixed  $k \in \mathbb{N}$ , the resulting finite difference scheme is again consistent to the incompressible Navier-Stokes system but now with Reynolds number  $Re = 6/k$ . This is easily seen by replacing  $\Delta x$  and  $\Delta t$  with  $k\Delta x$  and  $k\Delta t$  in (44) and (45). Obviously,  $k$  factors out in all approximations of first order derivatives but remains in front of the second order ones. Note that the corresponding finite difference stencils have width  $2k + 1$ .

## 7 LBM as combination of finite difference schemes

Writing the LB evolution (29) in operator form

$$f^{n+1} = \mathcal{L}_{\epsilon,\omega} f^n \quad (46)$$

we find that  $\mathcal{L}_{\epsilon,\omega}$  is a linear combination of two cases  $\mathcal{L}_{\epsilon,\omega} = (1 - \omega)\mathcal{L}_{\epsilon,0} + \omega\mathcal{L}_{\epsilon,1}$  where  $\mathcal{L}_{\epsilon,0}$  is defined as

$$(\mathcal{L}_{\epsilon,0}g)(\mathbf{x}, \mathbf{v}) = g(\mathbf{x} - \epsilon\mathbf{v}, \mathbf{v}) \quad (47)$$

and  $\mathcal{L}_{\epsilon,1}$  is related to the case  $\omega = 1$  which has been investigated in the previous section

$$\begin{aligned} (\mathcal{L}_{\epsilon,1}g)(\mathbf{x}, \mathbf{w}) &= f^{eq}(\rho_g(\mathbf{x} - \epsilon\mathbf{w}), \epsilon\mathbf{u}_g(\mathbf{x} - \epsilon\mathbf{w}); \mathbf{w}), \\ \rho_g &= \langle g \rangle, \quad \mathbf{u}_g = \langle \mathbf{v}g \rangle / (\epsilon\rho_g). \end{aligned} \quad (48)$$

Note that  $\mathcal{L}_{\epsilon,0}$  is precisely the solution operator of the collision-less Boltzmann equation in the diffusion scaling  $\partial_t g + \mathbf{v} \cdot \nabla g / \epsilon = 0$ , i.e.  $\mathcal{L}_{\epsilon,0}g$  yields the solution of the transport equation at time  $\Delta t = \epsilon^2$ . The corresponding moments  $\rho$ ,  $\mathbf{u}$  and  $\Theta$  approximate equations (24) and (25) with  $\omega = 0$ . Note that, in the limit  $\epsilon \rightarrow 0$ , the Navier-Stokes equation are *not* obtained since  $\Theta$  is no longer related to  $\rho\mathbf{u} \otimes \mathbf{u}$ . Consequently, the lattice Boltzmann evolution (29) can be viewed as a linear combination of two schemes with only one of them being consistent to the incompressible Navier-Stokes system. Assuming that  $\tau > 1/2$  (which is reasonable in view of the Reynolds number (12)), the factor  $(1 - \omega)$  in front of the inconsistent part  $\mathcal{L}_{\epsilon,0}$  satisfies  $|1 - \omega| = |1 - 1/\tau| < 1$ .

Iterating (46), the  $n$ -th step of the lattice Boltzmann method is determined by  $\mathcal{L}_{\epsilon,\omega}^n$  which can be expressed in terms of a sum of all  $n$ -fold products of the building blocks  $\mathcal{L}_{\epsilon,0}$  and  $\mathcal{L}_{\epsilon,1}$

$$\mathcal{L}_{\epsilon,\omega}^n = \sum_{\alpha \in \{0,1\}^n} (1 - \omega)^{|\bar{\alpha}|} \omega^{|\alpha|} \mathcal{L}_{\epsilon,\alpha_n} \cdots \mathcal{L}_{\epsilon,\alpha_1}.$$

Here,  $\bar{\alpha}$  defined by  $\bar{\alpha}_i = 1 - \alpha_i$  is the complementary vector to  $\alpha$ . The modulus  $|\alpha| = \sum_{i=1}^n \alpha_i$  counts the number of one-components of  $\alpha$  so that  $|\bar{\alpha}|$  is the number of free flow steps  $\mathcal{L}_{\epsilon,0}$  in the  $n$ -fold product. Since  $|1 - \omega| < 1$ , the influence of terms with many free flow steps is small due to the damping by  $(1 - \omega)^{|\bar{\alpha}|}$ . In particular, if  $\omega > 1$ , the factors  $\omega^{|\alpha|}$  are amplifying so that the behavior of LBM is dominated by the kinetic scheme. In this case, the free transport part, which is not consistent to the Navier-Stokes equation, only acts as a correction. We remark that, in typical applications, LBM exactly operates in the range  $1/2 < \tau \leq 1$  which is related to  $1 \leq \omega < 2$ . The simplest situation, apart from  $\omega = 1$ , is given by  $\omega = 1 + C\epsilon$ . In this case, the influence of the pure free transport contribution is damped below the truncation error already after two steps. We have

$$\mathcal{L}_{\epsilon,\omega}^2 = \omega^2 \mathcal{L}_{\epsilon,1}^2 + (1 - \omega)\omega \mathcal{L}_{\epsilon,0} \mathcal{L}_{\epsilon,1} + (1 - \omega)\omega \mathcal{L}_{\epsilon,1} \mathcal{L}_{\epsilon,0} + (1 - \omega)^2 \mathcal{L}_{\epsilon,0}^2. \quad (49)$$

Without going into details, we mention that for distribution functions  $g$  which are consistent to our scaling in the sense that  $Q_i$  moments of  $g$  are order one quantities,

$$(\mathcal{L}_{\epsilon,0}^2 - \mathcal{L}_{\epsilon,1}^2)g = \langle \mathcal{O} \rangle(\epsilon^2), \quad (\mathcal{L}_{\epsilon,0}\mathcal{L}_{\epsilon,1} - \mathcal{L}_{\epsilon,1}\mathcal{L}_{\epsilon,0})g = \langle \mathcal{O} \rangle(\epsilon^3) \quad (50)$$

where  $\langle \mathcal{O} \rangle(\epsilon^k)$  means that values of order  $\epsilon^k$  are obtained after integration with weights  $Q_0, Q_1$  and  $Q_2$ . Using (50) in connection with (49), taking into account that  $(1 - \omega) = \mathcal{O}(\epsilon)$  and setting  $\lambda = 2(1 - \omega)\omega$ , we obtain

$$\mathcal{L}_{\epsilon,\omega}^2 g = (1 - \lambda)\mathcal{L}_{\epsilon,1}^2 g + \lambda\mathcal{L}_{\epsilon,0}\mathcal{L}_{\epsilon,1}g + \langle \mathcal{O} \rangle(\epsilon^4). \quad (51)$$

Due to our remark at the end of Section 6

$$(\mathcal{L}_{\epsilon,0}\mathcal{L}_{\epsilon,1}g)(\mathbf{x}, \mathbf{v}) = f^{eq}(\rho_g(\mathbf{x} - 2\epsilon\mathbf{v}), \epsilon\mathbf{u}_g(\mathbf{x} - 2\epsilon\mathbf{v}); \mathbf{v})$$

yields, after taking mass and momentum integrals, a finite difference scheme (formulated just in terms of  $\rho$  and  $\mathbf{u}$ ) which is consistent to the Navier-Stokes equation with Reynolds number  $Re = 6/2 = 3$ . On the other hand,  $\mathcal{L}_{\epsilon,1}^2$  corresponds to  $Re = 6$  so that the combination (51) describes, in relevant order, a finite difference scheme for the Navier-Stokes equation with

$$\frac{1}{Re} = \frac{1 - \lambda}{6} + \frac{\lambda}{3} = \frac{1}{3} \left( \frac{3}{2} - \omega \right) + \mathcal{O}(\epsilon^2) = \frac{1}{3} \left( \tau - \frac{1}{2} \right) + \mathcal{O}(\epsilon^2)$$

(since  $\tau = 1/(1 - (1 - \omega)) = 1 + (1 - \omega) + \mathcal{O}(\epsilon^2)$ ). This is the known relation (12) for LBM up to terms of the order  $\epsilon^2$ .

Relation (51) extends the result of Section 6 and leads to the following interpretation of LBM: For  $\omega = 1$ , LBM is an explicit finite difference scheme in the flow variables  $\rho$  and  $\mathbf{u}$ . For  $\omega = 1 + \mathcal{O}(\epsilon)$ , two steps of LBM can be identified (in the relevant  $\epsilon$ -order) with the combination of two finite difference schemes which are formulated in terms of  $\rho$  and  $\mathbf{u}$ . Both schemes are consistent to the Navier-Stokes equation but with different Reynolds numbers. By combining the schemes, a whole range of Reynolds numbers is covered and the combination parameter is directly related to the viscosity.

For  $\omega = 1 + \mathcal{O}(\epsilon^{\frac{1}{m}})$ , the influence of the pure free transport contribution is damped below the truncation error after  $m$  steps and an interpretation of LBM as combination of finite difference schemes with different but fixed Reynolds numbers is again possible but turns out to be quite complicated.

## 8 LBM as relaxation scheme

In the generic case  $\omega \neq 1$ , equation (30) for  $M_i^{n+1}$  involves contributions

$$\frac{1}{\langle Q_j^2 f^* \rangle} \left\langle Q_i Q_j \left( (1 - \omega)M_j^n + \omega M_j^{n,eq} \right) (\mathbf{x} - \mathbf{v}\Delta x) f^* \right\rangle, \quad j = 0, \dots, 8 \quad (52)$$

from all moments since

$$(1 - \omega)\mathbf{M} + \omega\mathbf{M}^{eq} = (\rho, \rho u_1, \rho u_2, \Theta_{11}^\omega, \Theta_{12}^\omega, \Theta_{22}^\omega, (1 - \omega)q_1, (1 - \omega)q_2, (1 - \omega)s)^T$$

with  $\Theta^\omega = (1 - \omega)\Theta + \omega\rho\mathbf{u} \otimes \mathbf{u}$ . According to Section 5, expression (52) represents finite difference approximations. Moreover, since LBM is  $\epsilon^2$ -consistent to the moment equations (24), (25), we can easily find out which of the stencils appears in relevant order. In fact, stencils are important in the update rule for  $M_i$  if they are related to expressions in the corresponding equation of the moment system. For example, the mass conservation in (24) does not involve  $\Theta^\omega, \mathbf{q}$  or  $s$  from which we can conclude that the stencils in (52) with  $i = 0$  and  $j = 3, \dots, 8$  lead to expressions which are of the order of the truncation error. The stencils resulting from  $j = 0, 1, 2$ , on the other hand, have already been introduced in Section 6. As in (44), we find

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + D_k \rho^n u_k^n = \frac{1}{6} L \rho^n + \mathcal{O}(\Delta t).$$

Introducing the pressure  $p^n$  through  $\rho^n = \bar{\rho}(1 + 3\epsilon^2 p^n)$ , we again conclude that incompressibility is approximately obtained by a pseudo compressibility method. In leading order, the LB-evolution in stencil form collapses to

$$\frac{1}{12\Delta x} \left( \begin{bmatrix} -1 & 0 & 1 \\ -4 & 0 & 4 \\ -1 & 0 & 1 \end{bmatrix} u_1^n + \begin{bmatrix} 1 & 4 & 1 \\ 0 & 0 & 1 \\ -1 & -4 & -1 \end{bmatrix} u_2^n \right) = \mathcal{O}(\Delta t) \quad (53)$$

reflecting  $\text{div } \mathbf{u} = 0$ . Similarly, the relevant stencils in the momentum equation are those given in (45). For the first component of  $\mathbf{u}$ , we find after replacing  $\rho\mathbf{u}$  in leading order by  $\bar{\rho}\mathbf{u}$  and dividing by  $\bar{\rho}$

$$\begin{aligned} & \frac{u_1^{n+1} - u_1^n}{\Delta t} + \frac{1}{12\Delta x} \begin{bmatrix} -1 & 0 & 1 \\ -4 & 0 & 4 \\ -1 & 0 & 1 \end{bmatrix} u_1^n u_1^n + \frac{1}{4\Delta x} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix} u_1^n u_2^n + \frac{1}{3} \frac{1}{12\Delta x} \begin{bmatrix} -1 & 0 & 1 \\ -4 & 0 & 4 \\ -1 & 0 & 1 \end{bmatrix} p^n \\ & = \left( \frac{1}{36\Delta x^2} \begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} + \frac{1/\omega - 1}{216\Delta x^2} \begin{bmatrix} 5 & 0 & 8 & 0 & 5 \\ 4 & 0 & -8 & 0 & 4 \\ 0 & 0 & -36 & 0 & 0 \\ 4 & 0 & -8 & 0 & 4 \\ 5 & 0 & 8 & 0 & 5 \end{bmatrix} \right) u_1^n + \mathcal{O}(\Delta t) \quad (54) \end{aligned}$$

A similar expression holds for the second component  $u_2$ . Note that the stencil for  $\Delta\mathbf{u}$  consists of two parts – one reflecting a numerical diffusion term, the other one a discretization of  $\text{div } S[\mathbf{u}]$ . The divergence in this expression is the same as for the tensor  $\mathbf{u} \otimes \mathbf{u}$  and the stencils of the components of  $S_{kl}[\mathbf{u}] = (\partial u_k / \partial x_l + \partial u_l / \partial x_k) / 2$  are obtained by analyzing (52) with  $i = 3, 4, 5$  and  $j = 1, 2$ . It turns out that

$$S[\mathbf{u}] \approx \begin{pmatrix} D_1 u_1 & \frac{1}{2}(\tilde{D}_2 u_1 + \tilde{D}_1 u_2) \\ \frac{1}{2}(\tilde{D}_2 u_1 + \tilde{D}_1 u_2) & D_2 u_2 \end{pmatrix}$$

where  $\tilde{D}_k$  are given in (36), (37) and  $D_k$  in (34) and (35). The combination of these stencils with those from the divergence lead to the 25-points stencil. In [30] it has been shown that the use of large stencils for the viscous Laplacian (like the

one appearing naturally in the lattice Boltzmann method) helps to avoid pressure oscillations which can occur in Navier-Stokes calculations on non-staggered grids.

As we have noted already in Section 4, the moment system is a relaxation-type system for the incompressible Navier-Stokes equation. Since the lattice Boltzmann method is equivalent to the above explicit finite difference approximation of the moment system, LBM can be regarded as a relaxation scheme. Note, however, that relaxation schemes are usually formulated in such a way that they turn into schemes for the limiting equations if the relaxation parameter is set to zero which is achieved by an implicit treatment of the stiff terms (see [17]). In the case of LBM, however, all terms are treated explicitly which forces  $\Delta t$  to be of order  $\epsilon^2$ . In particular, setting  $\epsilon$  to zero cannot lead to a new scheme. For a different approach based on the lattice Boltzmann moment system, see [16, 30].

Since the lattice Boltzmann evolution is (up to additional finite difference terms which are of order  $\Delta t$  or less) equivalent to the discretizations (53) and (54), it is not surprising that numerical stability problems can occur. In fact, the term  $\text{div } \mathbf{u} \otimes \mathbf{u}$  is discretized with central differences and it is well known [26] that this is unstable without appropriate diffusion. If the physical viscosity becomes small (high Reynolds number flows), the standard approach is to replace central differences in  $\text{div } \mathbf{u} \otimes \mathbf{u}$  by upwind discretizations to provide numerical diffusion. Since there is no such mechanism in the lattice Boltzmann algorithm, instabilities naturally occur for  $\omega$  close to two. Note that for  $\omega = 2$ , the 25-points Laplacian compensates the nine-point discretizations up to terms of order  $\mathcal{O}(\Delta x^2)$ . In view of a modified equation analysis, one expects that the viscosity has to be at least of order  $\Delta t = \epsilon^2$  to yield a stable scheme. Since the physical viscosity is the only source of viscosity in LBM, this leads to a restriction on the Reynolds number  $Re = \mathcal{O}(1/\epsilon^2)$ . In a calculation with  $\Delta x = \epsilon = 0.01$ , the maximal Reynolds number is thus of order  $1/\epsilon^2 = 10^4$  while a Reynolds number of order  $1/\epsilon^3 = 10^6$  will probably lead to numerical instabilities due to a lack of viscosity.

## 9 Conclusions

We have investigated a frequently used lattice Boltzmann method for incompressible Navier-Stokes flows in two dimensions. The results are also applicable to other lattice Boltzmann models because they rest on the basic observation that discrete microscopic transport together with weighted velocity averages is a reformulation of finite difference approximations. The investigation of the particular nine-velocity model implies:

- Standard LBM can be considered as a linear combination of a direct and a relaxation scheme. The discretization is based on fully explicit finite difference approximations.

- LBM can also be regarded as a linear combination of finite difference schemes for the Navier-Stokes equation with fixed viscosities.
- The kinetic aspects of LBM determine the structure of the finite difference stencils (transport+averaging=stencil).
- The kinetic formulation leads to an efficient evaluation of the stencils with comparably little communication.
- The scheme works in the stability constellation  $\Delta t/\Delta x^2 = \mathcal{O}(1)$  known from the explicit scheme for advection diffusion equations.
- Approximate incompressibility is assured by some pseudo-compressibility approach.
- For relaxation parameters close to one, the extra information in the kinetic variables is partly redundant since all particle densities are determined by  $\rho$  and  $\mathbf{u}$  (up to corrections of the order of the truncation error).
- The formulation as relaxation system shows the restricted relevance of some of the variables.
- The often claimed advantage of LBM, to be a simple algorithm which is easily parallelizable, is explained by the interpretation as fully explicit, low order finite difference method.

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