A Kinetic Approach to Hyperbolic Systems and the Role of Higher Order Entropies

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Abstract. The reformulation of conservation laws in terms of kinetic equations, which parallels the relation between Boltzmann and Euler equation, has been successfully used in the form of kinetic schemes. The central problem in the kinetic approach is the construction of suitable equilibrium distributions which generalize the Maxwellian in the Boltzmann–Euler case. Here, we present a solution to this problem which allows the construction of equilibrium distributions for general systems of hyperbolic conservation laws. The approach leads to the notion of higher order entropies and generalizes several approaches discussed by other authors.

1. Introduction

In order to explain the kinetic approach, we consider a simple advection process which can be described by the scalar, linear conservation law

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0, \qquad u(0,x) = u^0(x), \qquad x \in \mathbb{R}, t \geq 0 \tag{1}$$

where $a \in \mathbb{R}$ is a given constant (the advection velocity). It is well known that the solution of (1) is

$$u(t,x) = u^0(x - at). \tag{2}$$

An alternative model is given by a *kinetic approach*: a continuum of particles is distributed in such a way that the initial density u^0 is recovered. To obtain the correct evolution, each particle is given the velocity a and free movement is assumed. If the density of particles with velocity v at position x and time t is described by the function $f(t,x,v)$, the evolution is given by the *kinetic transport process*

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0, \qquad f(0,x,v) = u^0(x)\delta(v - a) \tag{3}$$

The relation between the conservation law (1) and the kinetic equation (3) is obtained through the initial value in the kinetic model which is based on the *constraint function* $\mu(u;v) = u\delta(v-a)$. We will see below, that for more general scalar conservation laws, other constraint functions have to be used and if discontinuities occur in the solution, the kinetic model has to be modified by a source term.

However, the relation between the solution of the kinetic model and the one of the conservation law is generally obtained in the following way (the symbol $\langle\cdot,\cdot\rangle_v$ denotes v–integration)

$$u(t,x) = \langle f(t,x,v)\rangle_v\,, \qquad f(t,x,v) = \mu(u(t,x);v)$$

For the advection equation this is easily checked because the solution of (3) is $f(t,x,v) = u^0(x-vt)\delta(v-a)$, so that $\langle f(t,x,v),1\rangle_v = u^0(x-at)$ which is (2). On the other hand, $\mu(u(t,x);v) = u^0(x-at)\delta(v-a) = u^0(x-vt)\delta(v-a)$.

Let us now turn to the Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0, \qquad u(0,x) = u^0(x), \qquad x\in\mathbb{R} \tag{4}$$

as a more complicated example. According to [13], the problem to find the entropy solution of (4) can be restated as finding a solution $f(t,x,v)$ of the transport equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = \frac{\partial m}{\partial v} \qquad \text{in } \mathscr{D}'(\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}) \tag{5}$$

where m is a non–negative bounded measure which is chosen to ensure a particular v–dependence of f

$$f(t,x,v) = \mu(u(t,x);v) \qquad \text{for some function } u(t,x). \tag{6}$$

Here, μ is the difference of two Heaviside functions $\mu(u;v) = H(v) - H(v-u)$. The relation between (4) and (5), (6) is as follows (for details see [13]): if u is the entropy solution of (4) then $f(t,x,v) = \mu(u(t,x);v)$ solves (5) for some non–negative bounded measure m. Conversely, if f, m solve (5), (6) then the v–average $u = \langle f,1\rangle_v$ of f is the entropy solution of the Burgers equation.

The measure m which serves as a Lagrange multiplier to ensure the constraint $f = \mu$ has the interesting property that its (t,x) support is concentrated on the points of discontinuity of u. In other words, for smooth solutions of the conservation law, f automatically keeps the form μ and satisfies the evolution of free transport

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0, \qquad f(0,x,v) = \mu(u^0(x);v) \tag{7}$$

exactly as in our initial example. It is remarkable that the nonlinear behavior of the solution to (4) can be described by an extremely simple, linear particle dynamics. On the other hand, it is also clear that the simple free streaming leads to wrong results as soon as shocks appear in the solution. In fact, shocks are naturally connected to a deceleration of the flow (e.g. in the Burgers equation the shock speed is the average of the speeds to the left and to the right of the discontinuity) but this effect can not be captured with a model where the particles are not subject to any force. Hence, the "collision" term $\partial_v m$ is required to replace, for example, high particle velocities by the shock velocity. To illustrate these ideas, we calculate $\tilde{u} = \langle f,1\rangle_v$ based on the solution of (7)

$$\tilde{u}(t,x) = \left\langle \mu(u^0(x-vt);v),1\right\rangle_v \tag{8}$$

for different $t \geq 0$ with the initial value depicted in figure 1. Up to the time of the shock, (8) yields the correct solution (see figures 2 and 3). At later times however, a

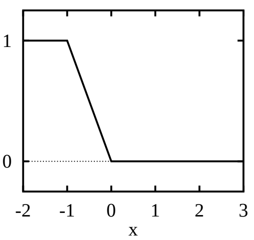

FIGURE 1. The initial value u^0

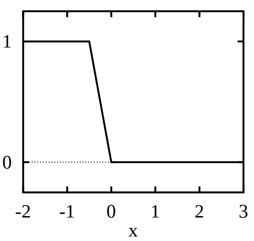

FIGURE 2. $u(0.5, x)$ and $\tilde{u}(0.5, x)$

rarefaction takes place which is no longer in accordance with the entropy solution of the Burgers equation and which results from the assumption of free flow underlying (8) (see figures 4, 5). Neglecting the source term $\partial_v m$ in (5) leads to a deviation of

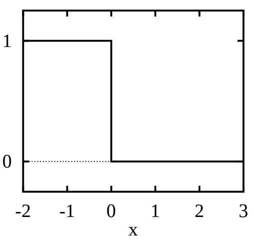

FIGURE 3. $u(1, x)$ and $\tilde{u}(1, x)$

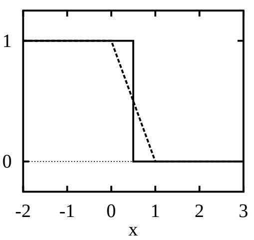

FIGURE 4. $u(2, x)$ and $\tilde{u}(2, x)$

the free flow solution $v \mapsto \mu(u^0(x - tv), v)$ from the form of the constraint function $v \mapsto \mu(\bar{u}; v)$. In fact, at time $t = 4$ the free flow solution at $x = 3/2$ is given in fig. 6. Note that it is still the difference of two Heaviside functions but no longer in the form $H(v) - H(v - 1/2)$ as for the exact solution. On the other hand, shortly after the shock time, the deviation of (8) from $u(t, x)$ is only small, i.e. we formally have first order consistency in time

$$u(\Delta t, x) = \langle f(\Delta t, x, v), 1 \rangle_v + \mathcal{O}(\Delta t^2)$$

where u solves (4) and f is the solution of the free flow equation (7). This observation can be used to derive approximate solutions of the conservation law and it is the basis of kinetic schemes.

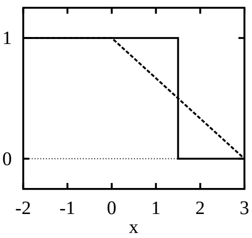

FIGURE 5. $u(4,x)$ and $\tilde{u}(4,x)$

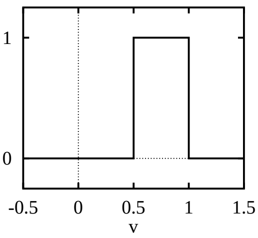

FIGURE 6. $\mu(u^0(\frac{3}{2} - 4v); v)$

2. Kinetic Schemes

Let us assume that a hyperbolic system of m equations in d dimensions

$$\frac{\partial}{\partial t}\boldsymbol{U} + \frac{\partial}{\partial x_j}\boldsymbol{F}^j(\boldsymbol{U}) = \boldsymbol{0}, \qquad \boldsymbol{U}(0,\boldsymbol{x}) = \boldsymbol{U}^0(\boldsymbol{x}). \tag{9}$$

is related to a system of m independent free flow equations

$$\frac{\partial}{\partial t}\boldsymbol{f} + v_j\frac{\partial}{\partial x_j}\boldsymbol{f} = \boldsymbol{0}, \qquad \boldsymbol{f}(0,\boldsymbol{x},\boldsymbol{v}) = \boldsymbol{\mu}(\boldsymbol{U}^0(\boldsymbol{x});\boldsymbol{v}) \tag{10}$$

in such a way that

$$\boldsymbol{U}(\Delta t,\boldsymbol{x}) = \langle \boldsymbol{f}(\Delta t,\boldsymbol{x},\boldsymbol{v}), 1\rangle_{\boldsymbol{v}} + \mathcal{O}(\Delta t^{n+1}) \tag{11}$$

for some $n \in \mathbb{N}$. (In the previous section, we have presented such relations for the case $m = 1$.) Then, kinetic schemes for (9) are generally constructed as follows:

a) Instead of (9), the *linear* equation (10) is discretized in $t, \boldsymbol{x}$ (finite difference, finite volume, finite element, etc.). The initial value for (10) at time step $n+1$ is based on the approximate value $\boldsymbol{U}^n$ resulting from the previous step.
b) The discrete evolution obtained in (a) is integrated over $\boldsymbol{v}$ to get a scheme for the *non–linear*, hyperbolic problem (9).

Schemes following this approach have been presented by several authors [1, 2, 4, 5, 6, 7, 12, 15, 16].

3. The central question

In order to apply the kinetic approach to a given hyperbolic system like (9), the key problem is to find a suitable constraint function $\boldsymbol{\mu}$ which relates the system to the kinetic model. In the following, we are trying to construct $\boldsymbol{\mu}$ in such a way that the order of consistency between solutions of (9) and (10) becomes maximal. In other words, we try to maximize n in the relation (11). In order to use Taylor expansion arguments, we restrict ourselves to *smooth* initial conditions and corresponding smooth solutions of the hyperbolic system. Note, however, that this restriction is only taken for the *construction* of $\boldsymbol{\mu}$. The obtained constraint function can, of course, be used in a kinetic scheme to approximate weak solutions of the hyperbolic system.

3.1. Assumptions on the hyperbolic system

We consider general hyperbolic problems

$$\frac{\partial}{\partial t}\boldsymbol{U}(t,\boldsymbol{x}) + \frac{\partial}{\partial x_j}\boldsymbol{F}^j(\boldsymbol{U}(t,\boldsymbol{x})) = \boldsymbol{0}, \qquad \boldsymbol{U}(0,\boldsymbol{x}) = \boldsymbol{U}^0(\boldsymbol{x}) \tag{12}$$

with $\boldsymbol{x} \in \mathbb{R}^d$ and assume that the unknowns $\boldsymbol{U} = (U_1, \ldots, U_m)^T$ are contained in a connected open set $\mathcal{S} \subset \mathbb{R}^m$ (the *state space*) with $\boldsymbol{F}^j : \mathcal{S} \mapsto \mathbb{R}^m$ being C^1–functions. In the generic case $d > 1$ and $m > 1$, we also assume that $\mathcal{S}$ is simply connected. Note that (12) is *hyperbolic* if all linear combinations $\xi_j A^j(\boldsymbol{U})$ of the Jacobian matrices $A^j(\boldsymbol{U}) = \nabla \boldsymbol{F}^j(\boldsymbol{U})$ of the fluxes have only real eigenvalues for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and all $\boldsymbol{U} \in \mathcal{S}$.

Concerning classical solutions of (12), we consider the spaces $\mathscr{I}_T^k$ of $\mathcal{S}$–valued functions $\boldsymbol{U} \in C^k([-T,T] \times \mathbb{R}^d, \mathcal{S})$ which have uniformly bounded derivatives and for which $\boldsymbol{U}([-T,T] \times \mathbb{R}^d)$ is a compact subset of $\mathcal{S}$. Using this notation, our assumption can be stated in the following way: for any $\boldsymbol{U}^0 \in \mathscr{I}_0^\infty$ there exists $T > 0$ such that (12) admits a classical solution $\boldsymbol{U} \in \mathscr{I}_T^1$.

3.2. Formulation of the problem

Given a hyperbolic system like (9), the central problem in the kinetic approach is the construction of a constraint function $\boldsymbol{\mu}$ such that equation i of the system

$$\frac{\partial U_i}{\partial t} + \frac{\partial}{\partial x_j} F_i^j(\boldsymbol{U}) = 0, \qquad U_i(0,\boldsymbol{x}) = U_i^0(\boldsymbol{x}) \tag{13}$$

is related to the kinetic equation

$$\frac{\partial f_i}{\partial t} + v_j \frac{\partial f_i}{\partial x_j} = 0, \qquad f_i(0,\boldsymbol{x},\boldsymbol{v}) = \mu_i(\boldsymbol{U}^0(\boldsymbol{x});\boldsymbol{v}).$$

Since the equations for f_i are decoupled, we can avoid dealing with systems by focusing on each equation (13) at a time. More generally, we can use the observation that $U_i = \eta(\boldsymbol{U})$ is a *linear entropy* for (12) with fluxes $\phi^j(\boldsymbol{U}) = F_i^j(\boldsymbol{U})$. The original problem is thus transformed into the question, how to relate an entropy conservation law for (12)

$$\frac{\partial}{\partial t}\eta(\boldsymbol{U}) + \frac{\partial}{\partial x_j}\phi^j(\boldsymbol{U}) = 0 \tag{14}$$

to the kinetic problem

$$\frac{\partial f}{\partial t} + v_j \frac{\partial f}{\partial x_j} = 0, \qquad f(0,\boldsymbol{x},\boldsymbol{v}) = \mu_\eta(\boldsymbol{U}^0(\boldsymbol{x});\boldsymbol{v})$$

Although this generalization is mainly a technical trick which reduces the original problem to m scalar ones, it will also give insight into the relation between entropy conservation laws and the kinetic approach.

Since the solution of (14) is given by

$$\tilde{\eta}(t,\boldsymbol{x}) = \left\langle \mu_\eta(\boldsymbol{U}^0(\boldsymbol{x}-t\boldsymbol{v});\boldsymbol{v})\right\rangle_{\boldsymbol{v}} \tag{15}$$

our aim is now to find μ_η such that

$$\eta(\boldsymbol{U}(\Delta t,\boldsymbol{x})) - \tilde{\eta}(\Delta t,\boldsymbol{x}) = \mathcal{O}(\Delta t^{n+1}) \tag{16}$$

with the order of consistency as large as possible.

In order to give (15) a precise mathematical meaning, we have to require some properties of μ_η. For fixed $\boldsymbol{U} \in \mathcal{S}$, we assume that $\mu_\eta(\boldsymbol{U})$ is a compactly supported distribution (we also write $\mu_\eta(\boldsymbol{U};\boldsymbol{v})$ to indicate that $\mu_\eta(\boldsymbol{U})$ acts in the $\boldsymbol{v}$–variable). Introducing as usual $\mathscr{E}(\mathbb{R}^d)$ as the space of C^∞ functions with a topology generated by the semi-norms

$$q_n(\varphi) = \max_{|\boldsymbol{\alpha}|\le n} \sup_{|\boldsymbol{v}|\le n} |\nabla^{\boldsymbol{\alpha}}\varphi(\boldsymbol{v})|, \qquad \varphi \in \mathscr{E}(\mathbb{R}^d),\ n \in \mathbb{N},$$

(we use standard multi–index notation) the compactly supported distributions $\mathscr{E}'(\mathbb{R}^d)$ are the continuous linear functionals on $\mathscr{E}$. Using this notation, we require that $\boldsymbol{U} \mapsto \mu_\eta(\boldsymbol{U})$ is a continuous mapping with values in $\mathscr{E}'$ which has some locally uniform properties.

Definition 3.1. *By $\mathscr{K}$ we denote the set of all continuous functions $\mu : \mathcal{S} \mapsto \mathscr{E}'(\mathbb{R}^d)$ which satisfy for any compact $K \subset \mathcal{S}$ and any $\varphi \in \mathscr{E}(\mathbb{R}^d)$*

$$|\left\langle \mu(\boldsymbol{U}), \varphi\right\rangle| \le C_K q_{N_K}(\varphi), \qquad \forall \boldsymbol{U} \in K$$

where N_K and C_K depend on μ and K. The subset $\mathscr{K}^1 \subset \mathscr{K}$ contains all μ for which there exists a continuous mapping $\nabla\mu : \mathcal{S} \mapsto [\mathscr{E}'(\mathbb{R}^d)]^m$ such that

$$\nabla\left\langle \mu(\boldsymbol{U}), \varphi\right\rangle = \left\langle \nabla\mu(\boldsymbol{U}), \varphi\right\rangle \qquad \forall \varphi \in \mathscr{E}(\mathbb{R}^d).$$

In [11], it is shown that for $\boldsymbol{U}^0 \in \mathscr{J}_0^0$, equation (15) defines a mapping $\eta \in C^\infty(\mathbb{R}, \mathscr{S}'(\mathbb{R}^d))$ where $\mathscr{S}'(\mathbb{R}^d)$ denotes the set of tempered distributions. On the other hand, if $\boldsymbol{U} \in \mathscr{J}_T^1$, the function $\boldsymbol{x} \mapsto \eta(\boldsymbol{U}(t,\boldsymbol{x}))$ is uniformly bounded for every $t \in [-T,T]$ so that it can also be viewed as a tempered distribution. A more precise formulation of (16) is based on the following

Definition 3.2. *Let $\eta \in C^1(\mathcal{S},\mathbb{R})$, $\mu \in \mathscr{K}$ and $n \in \mathbb{N}_0$. The constraint function μ is called n–consistent to η if for all $\boldsymbol{U}^0 \in \mathscr{J}_0^\infty$ with corresponding solution $\boldsymbol{U} \in \mathscr{J}_T^1$ of* (12)*, relation* (16) *holds in $\mathscr{S}'(\mathbb{R}^d)$, i.e. for all $\psi \in \mathscr{S}(\mathbb{R}^d)$*

$$\left\langle \eta(\boldsymbol{U}(\Delta t,\boldsymbol{x})) - \left\langle \mu(\boldsymbol{U}^0(\boldsymbol{x}-\boldsymbol{v}\Delta t);\boldsymbol{v}), 1\right\rangle_{\boldsymbol{v}}, \psi(\boldsymbol{x})\right\rangle_{\boldsymbol{x}} = \mathcal{O}(\Delta t^{n+1}).$$

The central problem can now be stated as follows: for a given hyperbolic system (12) and some $\eta \in C^1(\mathcal{S},\mathbb{R})$ find $\mu \in \mathscr{K}$ which is n–consistent to η with n as large as possible.

3.3. The result

The answer to the questions raised in the previous section is intimately related to the notion of *higher order entropies.* To introduce this concept, we need n–fold symmetric products $\boldsymbol{A}^{\boldsymbol{\alpha}}, |\boldsymbol{\alpha}| = n$ of the flux Jacobians $A^1, \dots A^d$ which are defined through the relation

$$\frac{1}{n!}(\xi_j A^j)^n = \sum_{|\boldsymbol{\alpha}|=n} \frac{1}{\boldsymbol{\alpha}!}\boldsymbol{\xi}^{\boldsymbol{\alpha}}\boldsymbol{A}^{\boldsymbol{\alpha}}. \tag{17}$$

For example, if $\boldsymbol{e}_i$ are the standard unit vectors, then $\boldsymbol{A}^{\boldsymbol{0}} = I$, $\boldsymbol{A}^{\boldsymbol{e}_i} = A^i$, and $\boldsymbol{A}^{\boldsymbol{e}_i+\boldsymbol{e}_j} = \frac{1}{2}(A^iA^j + A^jA^i)$.

Definition 3.3. *A function $\eta \in C^1(\mathcal{S}, \mathbb{R})$ is called entropy of order $n \in \mathbb{N}_0$ for the system* (12) *if the continuous mappings*

$$\boldsymbol{U} \mapsto \nabla^T\eta(\boldsymbol{U})\boldsymbol{A}^{\boldsymbol{\alpha}}(\boldsymbol{U})A^k(\boldsymbol{U}), \qquad 0 \le |\boldsymbol{\alpha}| < n,\ k = 1, \dots, d$$

have primitives.

We remark that entropies of order zero are just smooth functions on $\mathcal{S}$ (due to an empty assumption) and that usual entropies for (12) are recovered as first order entropies (the required primitives are then called entropy fluxes).

Theorem 3.4. *Assume the system* (12) *satisfies the conditions in Section 3.1 and $\eta \in C^1(\mathcal{S}, \mathbb{R})$. Then, there exists an n–consistent constraint function $\mu \in \mathscr{K}$ for η if and only if η is an entropy of order n.*

The required details for the proof can be found in [9, 11]. Here, we just mention that the sufficiency part of the proof is constructive. In fact, if η is an entropy of order n, we introduce the constraint function

$$\begin{aligned} \mu_\eta(\bar{\boldsymbol{U}}; \boldsymbol{v}) &= \int_{\boldsymbol{U}^*}^{\bar{\boldsymbol{U}}} \nabla^T\eta(\boldsymbol{U})E(\boldsymbol{U}; \boldsymbol{v})\, d\boldsymbol{U} + \eta(\boldsymbol{U}^*)\delta(\boldsymbol{v}) \\ E(\boldsymbol{U}; \boldsymbol{v}) &= \mathcal{F}_{\boldsymbol{\xi}}^{-1} \exp(-i\xi_j \nabla \boldsymbol{F}^j(\boldsymbol{U}))\Big|_{\boldsymbol{v}} \end{aligned} \tag{18}$$

where $\mathcal{F}_{\boldsymbol{\xi}}^{-1}$ denotes the inverse Fourier transform with respect to $\boldsymbol{\xi}$, and the line integral is carried out along a curve in the state space connecting $\bar{\boldsymbol{U}}$ with a fixed point $\boldsymbol{U}^* \in \mathcal{S}$. In this construction, the assumption of hyperbolicity is crucial to ensure that $E(\boldsymbol{U}; \boldsymbol{v})$ is a compactly supported distribution in $\boldsymbol{v}$ (the argument is based on the Paley–Wiener theorem). It can be shown that the ansatz (18) automatically leads to maximal order of consistency in the sense specified in Definition 3.2. A practical application to the 1D Euler system will be presented in the final section, other examples can be found in [9, 10]. It turns out that μ_η given by (18) is equivalent to the constraint functions of several kinetic approaches discussed in the literature.

Coming back to the problem of constructing kinetic schemes for a system of hyperbolic equations, we conclude that the crucial relation (11) can be obtained for general systems of hyperbolic equations at least with $n = 1$. The reason is that the linear functions $\eta_i(\boldsymbol{U}) = U_i$ are first order entropies (with fluxes $F_i^j(\boldsymbol{U})$) and hence $\mu_i = \mu_{\eta_i}$ constructed with (18) are one–consistent. Hence, the vector constraint function $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)^T$ gives rise to (11) with $n = 1$. In this way, one can also deduce the constraint functions of our two initial examples. For the advection equation, we observe that the inverse Fourier transform of $\exp(-i\xi a)$ is $E(u; v) = \delta(v - a)$ so that, in connection with the linear entropy $\eta(u) = u$, integration of $\eta' E$ from 0 to u yields $\mu(u; v) = u\delta(v-a)$. Similarly, for the Burgers equation, $E(u; v) = \delta(v-u)$ and the integration leads to $\mu(u; v) = H(v) - H(v-u)$ since $-H(v-u)$ is the u–primitive of $\delta(v-u)$.

3.4. Entropies of infinite order

A special situation occurs if all functions $\nabla^T\eta \boldsymbol{A}^{\boldsymbol{\alpha}} A^k$ have primitives, i.e. if η is of *infinite order*. It implies that for smooth solutions of the hyperbolic system, the entropy $\eta(\boldsymbol{U}(t, \boldsymbol{x}))$ coincides with the approximation obtained through the kinetic approach. To state the result more precisely, we introduce the notion of kinetic representations.

Definition 3.5. *Let $\eta \in C^1(\mathcal{S}, \mathbb{R})$ and $\mu \in \mathscr{K}$. We call μ a kinetic representation of η if for all $\boldsymbol{U}^0 \in \mathscr{J}_0^\infty$ with corresponding solution $\boldsymbol{U} \in \mathscr{J}_T^1$ of* (12)*, the equality*

$$\eta(\boldsymbol{U}(t, \boldsymbol{x})) = \left\langle \mu(\boldsymbol{U}^0(\boldsymbol{x} - t\boldsymbol{v}); \boldsymbol{v}), 1 \right\rangle_{\boldsymbol{v}}$$

holds in $\mathscr{S}'(R^d)$ for all $t \in [-T, T]$.

Theorem 3.6. *Assume the system* (12) *satisfies the conditions in Section 3.1 and $\eta \in C^1(\mathcal{S}, \mathbb{R})$. Then, η has a kinetic representation if and only if η is of infinite order.*

The sufficiency part of Theorem 3.6 also yields a complete characterization of the kinetic representation (see [11]).

Theorem 3.7. *Let η be of infinite order. Then there exists a kinetic representation $\mu \in \mathscr{K}^1$ with the property*

$$\nabla^T\mu(\boldsymbol{U}) = \nabla^T\eta(\boldsymbol{U})E(\boldsymbol{U}), \qquad E(\boldsymbol{U}) = \mathcal{F}_{\boldsymbol{\xi}}^{-1} \exp(-i\xi_j A^j(\boldsymbol{U})).$$

and $\langle \mu(\boldsymbol{U}), 1\rangle = \eta(\boldsymbol{U})$ for all $\boldsymbol{U} \in \mathcal{S}$. Any other kinetic representation differs from μ only by a compactly supported distribution $C \in \mathscr{E}'(\mathbb{R}^d)$ which is independent of $\boldsymbol{U}$ and satisfies $\langle C, 1\rangle = 0$.

It is easy to see that in the case of scalar conservation laws, every smooth function η is an entropy of infinite order (because primitives can always be obtained by integration). This shows that the kinetic approach is extremely well suited to the scalar case. For general linear hyperbolic systems, where $\boldsymbol{F}^j(\boldsymbol{U}) = A^j\boldsymbol{U}$ with constant matrices A^j, primitives of $\boldsymbol{A}^{\boldsymbol{\alpha}} A^k$ are just linear functions. Hence, at least all linear entropies are of infinite order in that case. For non–linear systems in higher dimensions, the assumption that entropies are of infinite order turns out to be quite restrictive. In fact, higher order entropies are more difficult to find than usual entropies which is not surprising since additional integrability conditions have to be satisfied. Non-trivial examples are obtained for the systems proposed by Brenier and Corrias [3] as well as the isentropic Euler equations with constant pressure.

3.5. Entropies of finite order

As example, we consider the Euler equations in one space dimension. Here, the vector of unknowns $\boldsymbol{U} = (\rho, \rho u, \rho\epsilon)^T$ consists of mass density ρ, momentum density ρu and energy density $\rho\epsilon$. Important derived quantities are velocity u, temperature $T = (\gamma-1)(\epsilon-u^2/2)$ and pressure $p = \rho T$ where $\gamma > 1$ is a material constant. The state space is a convex cone $\mathcal{S} = \{\, \rho(1, u, \epsilon)^T \mid \rho > 0, T > 0 \,\}$. The nonlinear flux $\boldsymbol{F}$ is homogeneous of degree one so that its Jacobian A is homogeneous of degree zero

$$\boldsymbol{F} = \begin{pmatrix} \rho u \\ \rho(u^2+T) \\ \rho(\epsilon+T)u \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma-3)u^2 & (3-\gamma)u & \gamma-1 \\ \frac{1}{2}(\gamma-2)u^3 - \frac{\gamma}{\gamma-1}Tu & (\frac{3}{2}-\gamma)u^2 + \frac{\gamma}{\gamma-1}T & \gamma u \end{pmatrix}.$$

By taking the $\boldsymbol{U}$–curl of the rows of $(A)^n$, it can be checked whether the linear entropies are of higher order. For the entropy $\eta_1(\boldsymbol{U}) = \rho$ one finds second order (third order if $\gamma = 3$), for $\eta_2(\boldsymbol{U}) = \rho u$ first order (second order if $\gamma = 3$), and for $\eta_3(\boldsymbol{U}) = \rho\epsilon$ always first order. Hence, the maximal order of consistency in (11) is restricted to $n = 1$. A constraint function can be constructed based on formula (18) with $\eta = \eta_i$ to determine the component μ_i. The choice of integration curves $\Gamma_{\boldsymbol{U}}$ is motivated by the structure of $\mathcal{S}$ and $\boldsymbol{F}$

$$\Gamma_{\boldsymbol{U}} := \{\, s\boldsymbol{U} \mid s \in (0,1] \,\} \qquad \boldsymbol{U} \in \mathcal{S}.$$

On these curves the Jacobian A is constant due to homogeneity of $\boldsymbol{F}$ so that

$$\boldsymbol{\mu}(\boldsymbol{U}; v) = \mathcal{F}_\xi^{-1} \exp(-i\xi A(\boldsymbol{U}))\boldsymbol{U}. \tag{19}$$

To calculate $\mathcal{F}_\xi^{-1} \exp(-i\xi A)$ we diagonalize A which has eigenvalues $\lambda_1 = u$, $\lambda_2 = u-c$ and $\lambda_3 = u+c$ with the sound speed $c = \sqrt{\gamma T}$. In a basis of right eigenvectors, the matrix $\exp(-i\xi A)$ has the form $\mathrm{diag}(\exp(-i\xi\lambda_k))$ so that the inverse Fourier transform yields a linear superposition of $\delta(v-\lambda_k)$. Using the abbreviation

$$f(\boldsymbol{U}; v) = \rho\,(2(\gamma-1)\delta(v-u) + \delta(v-u+c) + \delta(v-u-c))\,/2\gamma$$

the resulting constraint function can be written as

$$\boldsymbol{\mu}(\boldsymbol{U}; v) = (1, v, v^2/2 + (3-\gamma)/(2\gamma-2)|v-u|^2)^T f(\boldsymbol{U}; v).$$

We remark that the same constraint function follows from the approach in [8].

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