N-parabolic mixed order systems and free boundary value problems

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What is a mixed order system?

Roughly speaking this is matrix valued function $(\lambda, \xi) \in \mathbb{C} \times \mathbb{C}^n \mapsto \mathscr{L}(\lambda, \xi) \in \mathbb{C}^{m \times m}$ (also called *symbol*) which enables us to define an operator of the form

$$\begin{aligned} \mathscr{L}(\partial_t, \nabla) : \prod_{k=1}^{m} \mathbb{H}_k \quad &\longrightarrow \quad \prod_{k=1}^{m} \mathbb{F}_k, \\ (f_1, \dots, f_m)^T \quad &\mapsto \quad \mathcal{L}^{-1} \mathscr{F}^{-1} \mathscr{L}(\lambda, \xi) \mathscr{F} \mathcal{L}(f_1, \dots, f_m)^T \end{aligned}$$

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- How to find spaces such that $\mathscr{L}(\partial_t, \nabla) \in L(\prod_{k=1}^m \mathbb{H}_k, \prod_{k=1}^m \mathbb{F}_k)$?
- In which cases we have $\mathscr{L}(\partial_t, \nabla) \in L_{\text{Isom}}(\prod_{k=1}^m \mathbb{H}_k, \prod_{k=1}^m \mathbb{F}_k)$?

What are mixed order systems good for?

Two basic examples:

1) The thermo-elastic plate equations on $\Omega = \mathbb{R}^n$:

$$(\star) \begin{cases} u_{tt} + a \cdot \Delta^2 u + b \cdot \Delta\theta = f & ,(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \\ d \cdot \theta_t - g \cdot \Delta\theta - b \cdot \Delta u_t = g & ,(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \\ \gamma_0 u = 0, \gamma_0 u_t = 0, \gamma_0 \theta = 0 \end{cases}$$

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with $a, b, d, g > 0$. We can also write this problem with an formal Laplace and Fourier transform as

$$\mathcal{L}^{-1}\mathscr{F}^{-1}\begin{pmatrix} \lambda^2 + a|\xi|^4 & -b|\xi|^2 \\ b\lambda|\xi|^2 & d\lambda + g|\xi|^2 \end{pmatrix}\mathscr{F}\mathcal{L}\begin{pmatrix} u \\ \theta \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

In order to find spaces such that $(\star)$ is well-posed we can consider the mapping properties of the associated mixed oder system.

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2) The Stokes problem on $\Omega = \mathbb{R}^n$:

$$(\star\star)\begin{cases} \partial_t u - \Delta u + \nabla\pi &= f \quad ,(t,x)\in\mathbb{R}_+\times\mathbb{R}^n,\\ \operatorname{div} u &= g \quad ,(t,x)\in\mathbb{R}_+\times\mathbb{R}^n,\\ \gamma_0 u &= 0 \end{cases} \tag{1}$$

Formal Laplace and Fourier transform yields

$$\mathcal{L}^{-1}\mathscr{F}^{-1}\begin{pmatrix} \lambda+|\xi|^2 & i\xi \\ i\xi^T & 0 \end{pmatrix}\mathscr{F}\mathcal{L}\begin{pmatrix} u \\ \pi \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

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- These examples are interesting because our method is alternative to the usual semigroup approach.
- Later we see more serious applications to free boundary value problems!

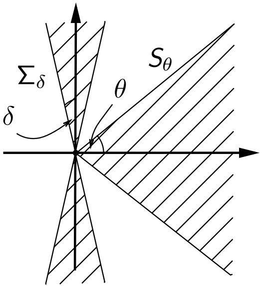
Contents

How to define $f(\partial_t, \nabla)$?

Outline: In the following we always assume $f \in H_P(S_\theta \times \Sigma_\delta^n)$ i.e

$$f : S_\theta \times \Sigma_\delta^n \longrightarrow \mathbb{C}, \qquad \theta \in (0, 2\pi), \delta \in (0, \pi/2)$$

is holomorphic and polynomially bounded.



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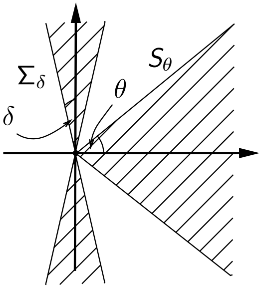

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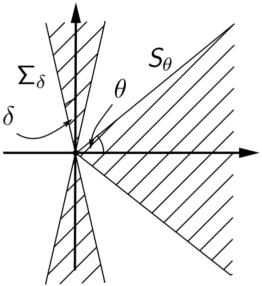

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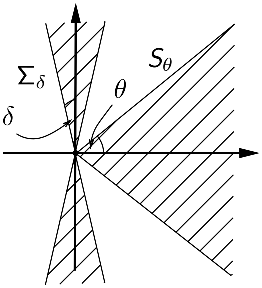
- ∂_t is a sectorial operator with angle $\theta = \pi/2$.
- G. Dore and A. Venni (2005) introduced the (joint) H^∞-calculus for sectorial and bisectorial operators
- ... and they proved that ∇ has a bounded joint H^∞-calculus on $L_q(\mathbb{R}_+, L_p(\mathbb{R}^n))$.
  $\rightsquigarrow$ representation by Fourier multiplier



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- Lifting by isomorphisms and interpolation yields that ∇ also has a bounded joint H^∞-calculus on

$$W^{s,r}_{\mathcal{F},\mathcal{K}}(\mathbb{R}_+ \times \mathbb{R}^n) := {}_0\mathcal{F}^s(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n)), \quad s \geq 0, r \in \mathbb{R}$$

with

$$\mathcal{F} \in \begin{cases} \{B_{p_0,q_0}\,,\ H_{p_0}\} & , s > 0 \\ \{H_{p_0}\} & , s = 0 \end{cases}, \qquad \mathcal{K} \in \{B_{p_1,q_1}\,,\ H_{p_1}\}, \quad p_i, q_i \in (1, \infty).$$

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- It is known that ∂_t even has an $\mathcal{R}$-bounded H^∞-calculus on $W^{s,r}_{\mathcal{F},\mathcal{K}}(\mathbb{R}_+ \times \mathbb{R}^n)$.

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- The Kalton-Weis Theorem (in a version of Dore and Venni (2005)) then yields the bounded joint H^∞-calculus of $\nabla_+ := (\partial_t, \nabla)$ on $W^{s,r}_{\mathcal{F},\mathcal{K}}(\mathbb{R}_+ \times \mathbb{R}^n)$.

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So we are able to define the operator

$$f(\nabla_+) : D(f(\nabla_+)) \subseteq W^{s,r}_{\mathcal{F},\mathcal{K}}(\mathbb{R}_+ \times \mathbb{R}^n) \to W^{s,r}_{\mathcal{F},\mathcal{K}}(\mathbb{R}_+ \times \mathbb{R}^n).$$

for $f \in H_P(S_\theta \times \Sigma^n_\delta)$ with $\theta > \pi/2$. If $f \in H^\infty(S_\theta \times \Sigma^n_\delta)$ then $f(\nabla_+)$ is bounded.

Disadvantage of the H^∞-calculus

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Already in easy cases as

$$f(z) := -\sum_{k=1}^{n} z^2, \quad g(\lambda, z) := \lambda + f(z) \quad (\lambda, z) \in S_\theta \times \Sigma_\delta^n$$

one is not able to show $D(f(\nabla)) = D(\Delta) = L_p(\mathbb{R}_+, H_p^2(\mathbb{R}^n))$ resp. $D(g(\nabla_+)) = {}_0H_p^1(\mathbb{R}_+, L_p(\mathbb{R}^n)) \cap L_p(\mathbb{R}_+, H_p^2(\mathbb{R}^n))$ by the definition of $D(f(\nabla))$ resp. $D(g(\nabla_+))$.

We can handle this difficulty with the concepts of the next section.

Contents

Motivation for the definition of order functions and weight functions

Which order has the symbol

$$P(\lambda, z) = z_1 z_n + 2z_1 + \sqrt{\lambda}|z|^{3/2} + \lambda^{3/4} z_n + \lambda^{3/4}\sqrt{z_1} + \lambda^{3/4}?$$

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To answer this we have to weight the time covariable λ. For the weight $\gamma > 0$ we define the γ-order

$$\begin{aligned} d_\gamma(P) &:= \max\left\{2, 1, \frac{3}{2} + \frac{1}{2}\gamma, 1 + \frac{3}{4}\gamma, \frac{1}{2} + \frac{3}{4}\gamma, \frac{3}{4}\gamma\right\} \\ &= \max\left\{2, \frac{3}{2} + \gamma\frac{1}{2}, 1 + \frac{3}{4}\gamma\right\}. \end{aligned}$$

Note that $d_\gamma(P)$ depends on the representation!

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Note that $d_\gamma(P)$ depends on the representation!

One can show that

$$|P(\lambda, z)| \leq C(1 + |z|^2 + |\lambda|^{1/2}|z|^{3/2} + |\lambda|^{3/4}|z| + |\lambda|^{3/4})$$

for all $(\lambda, z) \in S_\theta \times \Sigma_\delta^n$.

Definition of order functions and weight functions

Definition

A continuous function $\mathcal{O} : \mathbb{R}_+ \to \mathbb{R}$ is called **increasing order function** if there exist $M \in \mathbb{N}$, $m_\ell(\mathcal{O}), b_\ell(\mathcal{O}) \geq 0$ with $m_{\ell-1}(\mathcal{O}) \leq m_\ell(\mathcal{O})$, $b_{\ell-1}(\mathcal{O}) \geq b_\ell(\mathcal{O})$, $(\ell = 1, \ldots, M)$ and

$$\mathcal{O}(\gamma) = \max_{\ell=0,\ldots,M} \{b_\ell(\mathcal{O}) + \gamma \cdot m_\ell(\mathcal{O})\}, \quad \gamma > 0.$$

Definition

Let $\mathcal{O}$ be an increasing order function then we define the associated **weight function** by

$$\Xi_{\mathcal{O}}(\lambda, z) := \sum_{(s,r)\in\nu(\mathcal{O})} |\lambda|^s |z|^r, \quad (\lambda, z) \in S_\theta \times \Sigma_\delta^n$$

with $\nu(\mathcal{O}) := \{(m_\ell(\mathcal{O}), b_\ell(\mathcal{O})) : \ell = 0, \ldots, M\}$
$\cup\{(0,0), (\max_\ell m_\ell(\mathcal{O}), 0), (0, \max_\ell b_\ell(\mathcal{O}))\}$

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Definition

Let $\mathcal{O}$ be an increasing or decreasing order function and $Q \in H_P(S_\theta \times \Sigma_\delta^n)$. Then $\mathcal{O}$ is called an **upper (resp. lower) order function of Q** if there exist $C > 0$ and $\lambda_0 > 0$ with

$$|Q(\lambda, z)| \leq C \cdot \Xi_{\mathcal{O}}(\lambda, z), \quad (\lambda, z) \in S_\theta \times \Sigma_\delta^n, \quad |\lambda| \geq \lambda_0$$

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Definition

A symbol $Q \in H_P(S_\theta \times \Sigma_\delta^n)$ is called **N-parabolic** if there exists an increasing order function $\mathcal{O}(Q)$ which is an upper and lower order function of Q i.e. $|Q| \approx \Xi_{\mathcal{O}(Q)}$.

Example: The order function $\mathcal{O}(\gamma) := \max\{1, 1/2 \cdot \gamma\}$ is a lower and upper order function of $\omega(\lambda, z) := \sqrt{\lambda - \sum_{k=1}^{n} z^2}$.

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The next theorems show that these estimates by weight functions can help to understand the mapping properties of the operator $Q(\nabla_+)$.

Proposition (Mapping property I)

Let $Q \in H_P(S_\theta \times \Sigma_\delta^n)$ *be arbitrary with upper increasing order function* $\mathcal{O}$ *and* $\mathcal{W} := W^{s',r'}_{\mathcal{F},\mathcal{K}}(\mathbb{R}_+ \times \mathbb{R}^n)$ *then we get for all* $\mu \geq \lambda_0(Q, \mathcal{O})$

$$D(Q_\mu(\nabla_+^{\mathcal{W}})) \supseteq \mathcal{V} := \bigcap_{\ell=0}^{M} W^{s'+m_\ell(\mathcal{O}),r'+b_\ell(\mathcal{O})}_{\mathcal{F},\mathcal{K}}(\mathbb{R}_+ \times \mathbb{R}^n)$$

with $Q_\mu := Q(\mu + \cdot, \cdot)$*. The restriction to* $\mathcal{V}$ *yields the bounded operator*

$$\left[Q_\mu(\nabla_+^{\mathcal{W}})\right]_{|\mathcal{V}} \in L(\mathcal{V}, \mathcal{W}).$$

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Proposition (Mapping property II)

Let $Q \in H_P(S_\theta \times \Sigma_\delta^n)$ be arbitrary with upper decreasing order function $\mathcal{O}$ and $\mathcal{W} := W^{s',r'}_{\mathcal{F},\mathcal{K}}(\mathbb{R}_+ \times \mathbb{R}^n)$ then we get for all $\mu \geq \lambda_0(Q,\mathcal{O})$

$$Q_\mu(\nabla_+^{\mathcal{W}}) \in L(\mathcal{W},\mathcal{V}), \quad \mathcal{V} := \bigcap_{\ell=0}^{M} W^{s'-m_\ell(\mathcal{O}),r'-b_\ell(\mathcal{O})}_{\mathcal{F},\mathcal{K}}(\mathbb{R}_+ \times \mathbb{R}^n).$$

Corollary

If $P \in H_P(S_\theta \times \Sigma_\delta^n)$ is an N-parabolic symbol then we even get

$$P_\mu(\nabla_+^{\mathcal{W}}) \in L_{Isom}\left(\bigcap_{\ell=0}^{M} W_{\mathcal{F},\mathcal{K}}^{s'+m_\ell(\mathcal{O}),r'+b_\ell(\mathcal{O})}(\mathbb{R}_+ \times \mathbb{R}^n), W_{\mathcal{F},\mathcal{K}}^{s',r'}(\mathbb{R}_+ \times \mathbb{R}^n)\right)$$

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$$P_\mu(\nabla_+^{\mathcal{W}}) \in L_{Isom}\left(\bigcap_{\ell=0}^{M} W_{\mathcal{F},\mathcal{K}}^{s'+m_\ell(\mathcal{O}),r'+b_\ell(\mathcal{O})}(\mathbb{R}_+ \times \mathbb{R}^n), W_{\mathcal{F},\mathcal{K}}^{s',r'}(\mathbb{R}_+ \times \mathbb{R}^n)\right)$$

for all $\mu \geq \lambda_0(P, \mathcal{O})$.

Example: We have seen before that $\omega(\lambda, z) := \sqrt{\lambda - \sum_{k=1}^n z^2}$ is N-parabolic with order $\mathcal{O}(\gamma) := \max\{1, 1/2 \cdot \gamma\}$. This yields

$$\begin{aligned}\omega_\mu(\nabla_+) \quad \in \quad & L_{\text{Isom}}\big({}_0B_q^{s'+1/2}(\mathbb{R}_+, H_p^{r'}(\mathbb{R}^n)) \cap {}_0B_q^{s'}(\mathbb{R}_+, H_p^{r'+1}(\mathbb{R}^n)), \\ & {}_0B_q^{s'}(\mathbb{R}_+, H_p^{r'}(\mathbb{R}^n))\big)\end{aligned}$$

for all $\mu \geq \lambda_0(\omega)$ and $s' > 0$, $r' \in \mathbb{R}$.

How to prove N-parabolicity of a symbol?

Let

$$P(\lambda, z) := \sum_{k=1}^{M} \varphi_k(\lambda, z)\lambda^{\beta_k} z^{\alpha_k}, \quad (\lambda, z) \in \overline{S}_\theta \times \overline{\Sigma}_\delta^n, \quad \beta_k \geq 0, \alpha_k \in \mathbb{N}_0^n$$

where φ_k is ρ-homogeneous of order N_k, holomorphic, and non-vanishing $(\Rightarrow |\varphi_k(\lambda, z)| \approx |\lambda|^{N_k/\rho} + |z|^{N_k})$.

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$$d_\gamma(P) := \max_{k=1\ldots,M} \{\gamma\beta_k + d_\gamma(\varphi_k) + |\alpha_k|\}, \quad d_\gamma(\varphi_k) := \max\{N_k, \gamma N_k/\rho\}, \quad \gamma > 0.$$

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With this we can define the γ-principal part

$$\pi_\gamma P(\lambda, z) = \sum_{k\in I(\gamma)} \pi_\gamma\varphi_k(\lambda, z)\lambda^{\beta_k} z^{\alpha_k}, \quad \pi_\gamma\varphi_k := \begin{cases} \varphi_k(0, \cdot), & \gamma < \rho \\ \varphi_k, & \gamma = \rho \\ \varphi_k(\cdot, 0), & \gamma > \rho \end{cases}$$

with $I(\gamma) := \{j : \gamma\beta_j + d_\gamma(\varphi_j) + |\alpha_j| = d_\gamma(P)\}$.

How to prove N-parabolicity of a symbol?

Theorem (K., 2009)

The (regular) symbol $P(\lambda, z) = \sum_{k=1}^{M} \varphi_k(\lambda, z)\lambda^{\beta_k} z^{\alpha_k}$ *is N-parabolic if and only if* P *has non-vanishing principal parts i.e.*

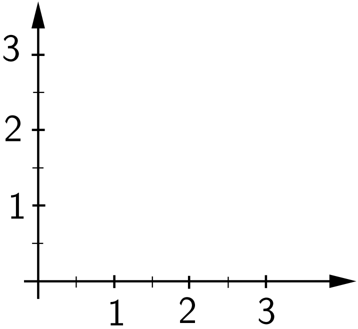
$$\pi_\gamma P(\lambda, z) \neq 0, \quad (\lambda, z) \in (\overline{S}_\theta \setminus \{0\}) \times (\overline{\Sigma}_\delta^n \setminus \{0\}), \quad \gamma > 0.$$

- 1992 this result was proved by S. Gindikin and L.R. Volevich for polynomials i.e. $\varphi_k \equiv 1$.
- 2008 R. Denk, J. Saal and J. Seiler proved the result for $n = 1$ and $\varphi_k = \omega^{N_k}$ with $\omega(\lambda, z) := \sqrt{\lambda - z^2}$.

Principal parts and Newton polygon

To determine the principal parts of a symbol we use the **Newton polygon** which is a useful tool to visualize the structure of inhomogeneous symbols.

$$P(\lambda, z) := \omega\lambda^2 - \omega z^2 - \lambda z^2 + \lambda z + 1$$



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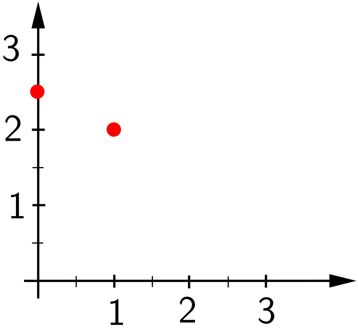

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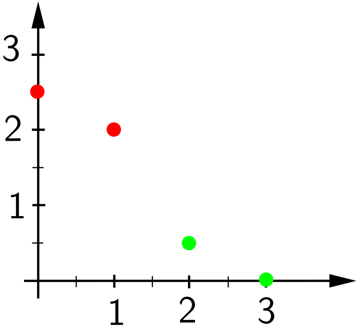

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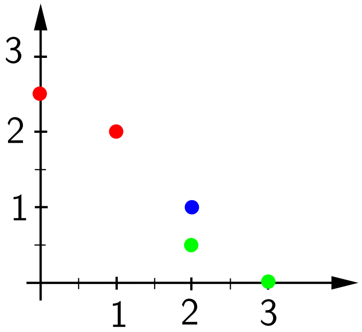

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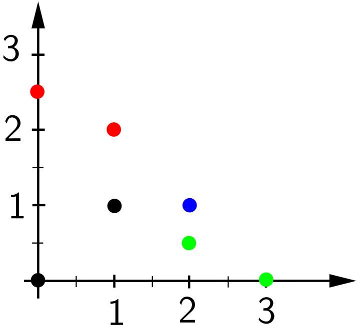

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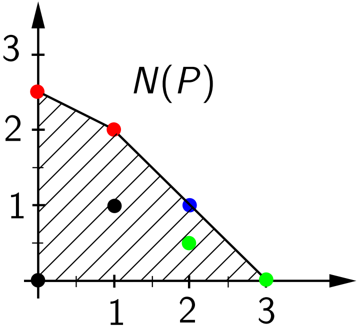

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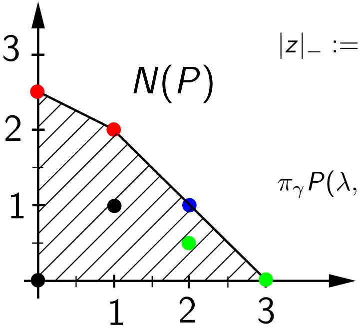

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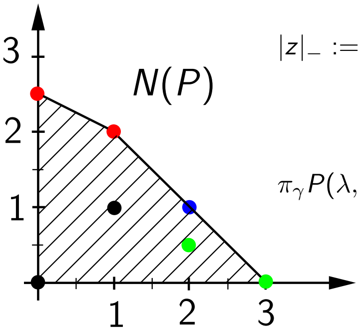

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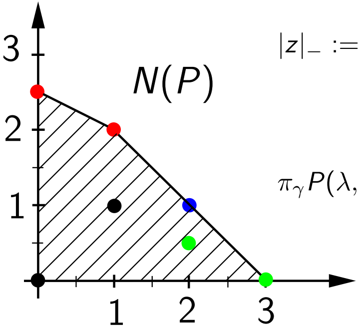

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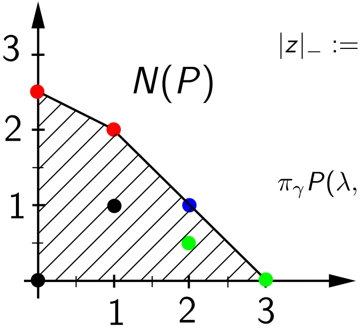

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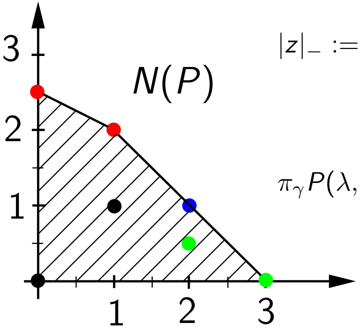

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Contents

Douglis-Nirenberg system

Definition

The system $\mathscr{L} \in [H_P(S_\theta \times \Sigma_\delta^n)]^{m\times m}$ is called a **mixed order system in the sense of Douglis-Nirenberg** if there are order functions $s_j := \mathcal{O}_j^{\text{row}}$ and $t_k := \mathcal{O}_k^{\text{col}}$ $(j,k = 1,\ldots,M)$ such that $s_j + t_k$ is an increasing or decreasing upper order function of $\mathscr{L}_{jk}$ for all $j,k = 1,\ldots,M$ (i.e. the upper order structure of each component splits into row and column part).

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Motivation for this Definition: With the help of the last section one can show that this property suffices to prove

$$[(\mathscr{L}_{jk})_\mu(\nabla_+)]_{|\mathbb{H}_k} \in L(\mathbb{H}_k, \mathbb{F}_j), \qquad j,k = 1,\dots,M$$

with

$$\begin{aligned} \mathbb{H}_k &:= \bigcap_{\ell=0}^{M} W_{\mathcal{F},\mathcal{K}}^{s'+m_\ell(t_k),r'+b_\ell(t_k)}(\mathbb{R}_+ \times \mathbb{R}^n), \\ \mathbb{F}_j &:= \bigcap_{\ell=0}^{M} W_{\mathcal{F},\mathcal{K}}^{s'-m_\ell(s_j),r'-b_\ell(s_j)}(\mathbb{R}_+ \times \mathbb{R}^n). \end{aligned}$$

N-parabolic mixed order system

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Let $\mathscr{L} \in [H_P(S_\theta \times \Sigma_\delta^n)]^{m\times m}$ be a mixed order system in the sense of Douglis-Nirenberg. Then the system $\mathscr{L}$ is called an **N-parabolic mixed order system** if

(i) $\det \mathscr{L}$ is N-parabolic,

(ii) $[\mathcal{O}(\det \mathscr{L})](\gamma) = \sum_{j=1}^m (s_j(\gamma) + t_j(\gamma))$ for all $\gamma > 0$.

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Motivation for this Definition: To simplify the notion we only consider the N-parabolic 2×2-system

$$\mathscr{L} = \begin{pmatrix} \mathscr{L}_{11} & \mathscr{L}_{12} \\ \mathscr{L}_{21} & \mathscr{L}_{22} \end{pmatrix} \in [H_P(S_\theta \times \Sigma_\delta^n)]^{2 \times 2}.$$

Hence we have

$$\mathscr{L}^{-1} = \begin{pmatrix} \frac{\mathscr{L}_{22}}{\mathscr{L}_{11}\mathscr{L}_{22} - \mathscr{L}_{12}\mathscr{L}_{21}} & \frac{-\mathscr{L}_{12}}{\mathscr{L}_{11}\mathscr{L}_{22} - \mathscr{L}_{12}\mathscr{L}_{21}} \\ \frac{-\mathscr{L}_{21}}{\mathscr{L}_{11}\mathscr{L}_{22} - \mathscr{L}_{12}\mathscr{L}_{21}} & \frac{\mathscr{L}_{11}}{\mathscr{L}_{11}\mathscr{L}_{22} - \mathscr{L}_{12}\mathscr{L}_{21}} \end{pmatrix}$$

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Therefore we can prove mapping properties for $\mathscr{L}^{-1}(\nabla_+)$ if $\mathscr{L}$ is an N-parabolic mixed order system.

N-parabolic mixed order systems can be found in works of R. Denk and L.R. Volevich (2008) (only L_2 but general weight functions)

Theorem (K., 2010)

Let $\mathscr{L} \in [H_P(S_\theta \times \Sigma_\delta^n)]^{m\times m}$ *be an N-parabolic mixed order system. We define for* $i,j = 1,\dots m$ *the spaces*

$$\mathbb{H}_i := \bigcap_{\ell=0}^{M} W^{s'_\ell+m_\ell(t_i),r'_\ell+b_\ell(t_i)}_{\mathcal{F}_\ell,\mathcal{K}_\ell}(\mathbb{R}_+\times\mathbb{R}^n), \quad \mathbb{F}_j := \bigcap_{\ell=0}^{M} W^{s'_\ell-m_\ell(s_j),r'_\ell-b_\ell(s_j)}_{\mathcal{F}_\ell,\mathcal{K}_\ell}(\mathbb{R}_+\times\mathbb{R}^n)$$

with fixed constants $s'_\ell, r'_\ell \geq 0$ *(*$\ell = 0,\dots,M$*) such that*

$$\begin{aligned}\mathcal{O}_{\mathbb{H}_i}(\gamma) &:= \max_\ell\{[s'_\ell+m_\ell(t_i)]\gamma + r'_\ell + b_\ell(t_i)\}, \quad \gamma>0, \quad i,j=1,\dots m\\ \mathcal{O}_{\mathbb{F}_j}(\gamma) &:= \max_\ell\{[s'_\ell-m_\ell(s_j)]\gamma + r'_\ell - b_\ell(s_j)\}, \quad \gamma>0, \quad i,j=1,\dots m\end{aligned}$$

are increasing order functions. If certain "compatibility embeddings" hold then there exists $\mu_0 > 0$ *such that for all* $\mu \geq \mu_0$

$$[\mathscr{L}_\mu(\nabla_+)]_{|\mathbb{H}} \in L_{Isom}(\mathbb{H},\mathbb{F}), \qquad ([\mathscr{L}_\mu(\nabla_+)]_{|\mathbb{H}})^{-1} = [\mathscr{L}_\mu^{-1}(\nabla_+)]_{|\mathbb{F}}$$

with $\mathbb{H} := \prod_{i=1}^m \mathbb{H}_i$ *and* $\mathbb{F} := \prod_{i=1}^m \mathbb{F}_i$*.*

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Two-phase Navier Stokes equation with surface tension

We consider the following linearized problem for $u = (v, w)$, π, and h:

$$\begin{cases} \rho\partial_t u - \mu\Delta u + \nabla\pi = 0 & \text{in } \mathbb{R}_+ \times \dot{\mathbb{R}}^{n+1} \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \dot{\mathbb{R}}^{n+1} \\ -[\![\mu\partial_y v]\!] - [\![\mu\nabla_x w]\!] = g_v & \text{on } \mathbb{R}_+ \times \mathbb{R}^n \\ -2[\![\mu\partial_y w]\!] + [\![\pi]\!] - \sigma\Delta h = g_w & \text{on } \mathbb{R}_+ \times \mathbb{R}^n \\ [\![u]\!] = 0 & \text{on } \mathbb{R}_+ \times \mathbb{R}^n \\ \partial_t h - \gamma_0 w = g_h & \text{on } \mathbb{R}_+ \times \mathbb{R}^n \\ u(t=0) = 0 & \text{in } \dot{\mathbb{R}}^{n+1} \\ h(t=0) = 0 & \text{in } \mathbb{R}^n \end{cases}$$

with $(u, \pi, \rho, \mu) = (u_1, \pi_1, \rho_1, \mu_1)\chi_{\mathbb{R}^{n+1}_-} + (u_2, \pi_2, \rho_2, \mu_2)\chi_{\mathbb{R}^{n+1}_+}$, $\sigma > 0$,
$[\![\phi]\!] := \gamma_0\phi_{\mathbb{R}^{n+1}_+} - \gamma_0\phi_{\mathbb{R}^{n+1}_-}$.

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with $(u, \pi, \rho, \mu) = (u_1, \pi_1, \rho_1, \mu_1)\chi_{\mathbb{R}^{n+1}_-} + (u_2, \pi_2, \rho_2, \mu_2)\chi_{\mathbb{R}^{n+1}_+}$, $\sigma > 0$,
$[\![\phi]\!] := \gamma_0\phi_{\mathbb{R}^{n+1}_+} - \gamma_0\phi_{\mathbb{R}^{n+1}_-}$.

After a formal application of the Laplace respectively Fourier transform we deduce a system of ODEs for $(\hat{u}, \hat{\pi})$.

Two-phase Navier Stokes equation with surface tension

Reduction to the boundary: We obtain

$$
\begin{aligned}
\mu_j\omega_j^2\hat{u}_j(\lambda,\xi,y) - \mu_j\partial_y^2\hat{u}_j(\lambda,\xi,y) + (i\xi,\partial_y)^T\hat{\pi}_j(\lambda,\xi,y) &= 0, \quad (-1)^j y > 0,\\
i\xi\cdot\hat{v}_j(\lambda,\xi,y) + \partial_y\hat{w}_j(\lambda,\xi,y) &= 0, \quad (-1)^j y > 0,\\
-[\![\mu\partial_y\hat{v}]\!](\lambda,\xi) - i\xi[\![\mu\hat{w}]\!](\lambda,\xi) &= \hat{g}_v(\lambda,\xi),\\
-2[\![\mu\partial_y\hat{w}]\!](\lambda,\xi) + [\![\hat{\pi}]\!](\lambda,\xi) + \sigma|\xi|^2\hat{h}(\lambda,\xi) &= \hat{g}_w(\lambda,\xi),\\
[\![\hat{u}]\!](\lambda,\xi) &= 0\\
\lambda\hat{h}(\lambda,\xi) - \hat{w}(\lambda,\xi,0) &= \hat{g}_h(\lambda,\xi),
\end{aligned}
$$

for fixed (λ,ξ), $\quad \omega_j = \omega_j(\lambda,\xi) := \mu_j^{-1/2}(\rho_j\lambda + \mu_j|\xi|^2)^{1/2}$.

Two-phase Navier Stokes equation with surface tension

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[\![\hat{u}]\!](\lambda,\xi) &= 0\\
\lambda\hat{h}(\lambda,\xi) - \hat{w}(\lambda,\xi,0) &= \hat{g}_h(\lambda,\xi),
\end{aligned}
$$

for fixed (λ,ξ), $\quad \omega_j = \omega_j(\lambda,\xi) := \mu_j^{-1/2}(\rho_j\lambda + \mu_j|\xi|^2)^{1/2}$.

Using Green's functions and $\mathrm{div} u = 0$ we get a solution of the ODE by ...

Two-phase Navier Stokes equation with surface tension

$$\begin{aligned}
\hat{\pi}_j(\lambda,\xi,y) &= \hat{p}_j(\lambda,\xi)\cdot\exp(-(-1)^j|\xi|y), \quad (-1)^j y>0,\\
\hat{v}_j(\lambda,\xi,y) &= -\int_0^\infty k_-^{(j)}(\lambda,\xi,s)\cdot i\xi\hat{\pi}_j(\lambda,\xi,s)ds+\hat{\Phi}_v^{(j)}(\lambda,\xi)\exp(-(-1)^j\omega_j y),\\
\hat{w}_j(\lambda,\xi,y) &= -\int_0^\infty k_+^{(j)}(\lambda,\xi,s)\cdot \partial_y\hat{\pi}_j(\lambda,\xi,s)ds+\hat{\Phi}_w^{(j)}(\lambda,\xi)\exp(-(-1)^j\omega_j y),
\end{aligned}$$

with unknown functions $\hat{\Phi}_v^{(j)}$, $\hat{\Phi}_w^{(j)}$, $\hat{p}_j$ and the Green's functions $k_\pm^{(j)}$

$$k_-^{(1)}(\lambda,\xi,y,s) := \begin{cases} -\frac{e^{\omega_1 y}}{\mu_1\omega_1}\sinh(\omega_1 s), & y\le s\\ -\frac{e^{\omega_1 s}}{\mu_1\omega_1}\sinh(\omega_1 y), & y\ge s \end{cases},$$

$$k_+^{(1)}(\lambda,\xi,y,s) := \begin{cases} \frac{e^{\omega_1 y}}{\mu_1\omega_1}\cosh(\omega_1 s), & y\le s\\ \frac{e^{\omega_1 s}}{\mu_1\omega_1}\cosh(\omega_1 y), & y\ge s \end{cases},$$

and $k_-^{(2)}=\dots$, $k_+^{(2)}=\dots$ analogously.

Two-phase Navier Stokes equation with surface tension

With this we derive

$$\begin{aligned}
\hat{\pi}_j(\lambda,\xi,0) &= \hat{p}_j(\lambda,\xi),\\
\hat{v}_j(\lambda,\xi,0) &= \hat{\Phi}_v^{(j)}(\lambda,\xi),\\
\hat{w}_j(\lambda,\xi,0) &= \frac{(-1)^j}{\mu_j}\cdot\frac{|\xi|}{\omega_j\gamma_j^+}\hat{p}_j(\lambda,\xi)+\hat{\Phi}_w^{(j)}(\lambda,\xi),\\
\partial_y\hat{v}_j(\lambda,\xi,0) &= -\frac{(-1)^j}{\mu_j}\frac{i\xi}{\gamma_j^+}\hat{p}_j(\lambda,\xi)-(-1)^j\omega_j\hat{\Phi}_v^{(j)}(\lambda,\xi),\\
\partial_y\hat{w}_j(\lambda,\xi,0) &= -(-1)^j\omega_j\hat{\Phi}_w^{(j)}(\lambda,\xi).
\end{aligned}$$

Plugging in this into the boundary conditions we obtain a system of linear equations for the unknowns $\hat{\Phi}_v^{(j)}$, $\hat{\Phi}_w^{(j)}$, $\hat{p}_2$, $[\![\hat{\pi}]\!]$, and $\hat{h}$.

Two-phase Navier Stokes equation with surface tension

$$\begin{pmatrix} i\xi^T & -\omega_2 & 0 & 0 & 0 & 0 \\ i\xi^T & 0 & \omega_1 & 0 & 0 & 0 \\ 0 & -1 & 0 & \lambda & 0 & -\frac{1}{\mu_2\omega_2\gamma_2^+} \\ 0 & 1 & -1 & 0 & -\frac{|\xi|}{\mu_1\omega_1\gamma_1^+} & \frac{1}{\mu_2\omega_2\gamma_2^+}+\frac{1}{\mu_1\omega_1\gamma_1^+} \\ 0 & 2\mu_2\omega_2 & 2\mu_1\omega_1 & \sigma|\xi|^2 & 1 & 0 \\ \Omega\mathrm{id}_n & -\mu_2 i\xi & \mu_1 i\xi & 0 & -i\xi\frac{\gamma_1^-}{\omega_1'\gamma_1^+} & \frac{i\xi}{|\xi|}\left[\frac{\gamma_2^-}{\omega_2\gamma_2^+}+\frac{\gamma_1^-}{\omega_1\gamma_1^+}\right] \end{pmatrix} \begin{pmatrix} \hat\Phi_v^{(2)} \\ \hat\Phi_w^{(2)} \\ \hat\Phi_w^{(1)} \\ \hat h \\ [\![\hat\pi]\!] \\ |\xi|\hat p_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \hat g_h \\ 0 \\ \hat g_w \\ \hat g_v \end{pmatrix}, \quad \Omega := \mu_1\omega_1 + \mu_2\omega_2, \quad \gamma_j^\pm := \omega_j \pm |\xi|.$$

The Theorems on mixed order systems do not cover homogeneous spaces so we have to eliminate the pressure $\hat p_2$.

Two-phase Navier Stokes equation with surface tension

Solving $[\![\hat{w}]\!] = 0$ we obtain the reduced system

$$\underbrace{\begin{pmatrix} i\xi^T & -\omega_2 & 0 & 0 & 0 \\ i\xi^T & 0 & \omega_1 & 0 & 0 \\ 0 & -\mu_2\omega_2\gamma_2^+\Omega_+^{-1} & -\mu_1\omega_1\gamma_1^+\Omega_+^{-1} & \lambda & -|\xi|\Omega_+^{-1} \\ 0 & 2\mu_2\omega_2 & 2\mu_1\omega_1 & \sigma|\xi|^2 & 1 \\ \Omega \mathrm{id}_n & -i\xi(\mu_2+\kappa) & i\xi(\mu_1+\kappa) & 0 & i\xi(\mu_2\gamma_2^- - \mu_1\gamma_1^-)\Omega_+^{-1} \end{pmatrix}}_{=:\mathscr{L}} \begin{pmatrix} \hat{\Phi}_v^{(2)} \\ \hat{\Phi}_w^{(2)} \\ \hat{\Phi}_w^{(1)} \\ \hat{h} \\ [\![\hat{\pi}]\!] \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \hat{g}_h \\ \hat{g}_w \\ \hat{g}_v \end{pmatrix}, \quad |\xi|\hat{p}_2 = -\delta\hat{\Phi}_w^{(2)} + \delta\hat{\Phi}_w^{(1)} + \frac{1}{\mu_1}\cdot\frac{\delta|\xi|}{\omega_1\gamma_1^+}[\![\hat{\pi}]\!]$$

with $\Omega_+ := \mu_1\omega_1\gamma_1^+ + \mu_2\omega_2\gamma_2^+$, $\delta := \frac{\mu_1\mu_2\omega_1\omega_2\gamma_1^+\gamma_2^+}{\Omega_+}$
$\kappa := \frac{\mu_1\mu_2}{|\xi|}\cdot(\omega_1\gamma_1^+\gamma_2^- + \omega_2\gamma_2^+\gamma_1^-)\Omega_+^{-1}$

Two-phase Navier Stokes equation with surface tension

As determinant we obtain $|\det \mathscr{L}| = |\omega_1\omega_2/\Omega_+| \cdot |\Omega^{n-1}| \cdot |P|$ with

$$\begin{aligned} P(\lambda, z) \quad := \quad & (\mu_1\omega_1^2 + \mu_2\omega_2^2)(\mu_1\omega_1 + \mu_2\omega_2)\lambda \\ & + \left[(\mu_1\omega_1 + \mu_2\omega_2)^2 + \mu_1\mu_2(\omega_1 + \omega_2)^2\right] \lambda|z|_- \\ & + \left[3(\mu_2^2\omega_2 + \mu_1^2\omega_1) - \mu_1\mu_2(\omega_1 + \omega_2)\right] \lambda|z|_-^2 \\ & -(\mu_1 - \mu_2)^2\lambda|z|_-^3 + \sigma(\mu_1\omega_1 + \mu_2\omega_2)|z|_-^3 \\ & +\bar{\mu}\sigma|z|_-^4 \end{aligned}$$

$\bar{\mu} := \mu_1 + \mu_2$, $\bar{\rho} := \rho_1 + \rho_2$, $|z|_- := \left(-\sum_{k=1}^n z_k^2\right)^{1/2} \in S_\delta$

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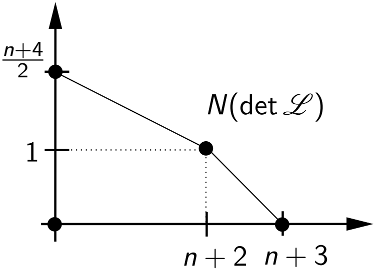
$$\pi_\gamma P(\lambda, z) = \begin{cases} 2\bar{\mu}\sigma|z|_-^4, & \gamma \in (0,1) \\ 2\bar{\mu}(2\bar{\mu}\lambda + \sigma|z|_-)|z|_-^3, & \gamma = 1 \\ 4\bar{\mu}^2\lambda|z|_-^3, & \gamma \in (1,2) \\ \bar{\rho}(\sqrt{\mu_1\rho_1} + \sqrt{\mu_2\rho_2})\lambda^{5/2}, & \gamma > 2 \end{cases}$$

$$\begin{aligned} \pi_{\gamma=2}P(\lambda, z) \quad = \quad & \bar{\rho}(\sqrt{\mu_1}\omega_1 + \sqrt{\mu_2}\omega_2)\lambda + 4\sqrt{\mu_1\mu_2}\omega_1\omega_2|z|_- + \bar{\rho}\bar{\mu}\lambda|z|_- \\ & +4(\mu_1^{3/2}\omega_1 + \mu_2^{3/2}\omega_2)|z|_-^2 + 4\mu_1\mu_2|z|_-^3. \end{aligned}$$

Two-phase Navier Stokes equation with surface tension

It is obvious that $\pi_\gamma P(\lambda, z) \neq 0$ for $\gamma \in (1, \infty) \setminus \{2\}$ and $(\lambda, z) \in (\overline{S}_\theta \setminus \{0\}) \times (\overline{\Sigma}_\delta^n \setminus \{0\})$. One can easily prove that even $\pi_{\gamma=2} P(\lambda, z) \neq 0$. Hence P (respectively $\det \mathscr{L}$) is N-parabolic with

$$[\mathcal{O}(\det \mathscr{L})](\gamma) = \max\{n+3, \gamma + n + 2, [n+4]/2\gamma\}, \quad \gamma > 0.$$



Therefore $\mathscr{L}$ is an N-parabolic mixed order system if we can find order functions s_j, t_j such that $s_j + t_k$ is an upper order function of $\mathscr{L}_{j,k}$ and $\mathcal{O}(\det \mathscr{L}) = \sum_{j=1}^n (s_j + t_j)$.

Two-phase Navier Stokes equation with surface tension

$$
\begin{aligned}
t_1(\gamma) := \ldots := t_{n+2}(\gamma) &:= \max\{1, \gamma/2\},\\
t_{n+3}(\gamma) &:= \max\{2, \gamma+1, 3/2\gamma\},\\
t_{n+4}(\gamma) &:= 0,\\
s_1(\gamma) := s_2(\gamma) &:= 0,\\
s_3(\gamma) &:= -\max\{1, \gamma/2\},\\
s_4(\gamma) := \ldots := s_{n+4}(\gamma) &:= 0
\end{aligned}
$$

$\Longrightarrow \mathcal{O}(\det \mathscr{L}) = \sum_{j=1}^{n}(s_j + t_j)$

Two-phase Navier Stokes equation with surface tension

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s_1(\gamma) := s_2(\gamma) &:= 0, \\
s_3(\gamma) &:= -\max\{1, \gamma/2\}, \\
s_4(\gamma) := \ldots := s_{n+4}(\gamma) &:= 0
\end{aligned}$$

$\Longrightarrow \mathcal{O}(\det \mathscr{L}) = \sum_{j=1}^{n}(s_j + t_j)$

$$\mathscr{L}(\lambda, z) \approx \begin{pmatrix} z^T & \omega & 0 & 0 & 0 \\ z^T & 0 & \omega & 0 & 0 \\ 0 & 1 & 1 & \lambda & -|z|_-/\omega^2 \\ 0 & \omega & \omega & |z|_-^2 & 1 \\ \omega \mathrm{id}_n & \omega & \omega & 0 & |z|_-/\omega \end{pmatrix}$$

Hence $\mathscr{L}$ is a Douglis-Nirenberg system with the order functions defined above.

Two-phase Navier Stokes equation with surface tension

So we can construct the spaces of maximal regularity with $(r_0', s_0') := (r_1', s_1') := (1 - 1/p, 0)$, $(r_2', s_2') := (0, 1/2 - 1/(2p))$, and $(\mathcal{F}_0, \mathcal{K}_0) := (\mathcal{F}_1, \mathcal{K}_1) = (H_p, B_p)$, $(\mathcal{F}_2, \mathcal{K}_2) = (B_p, H_p)$

$$\begin{aligned} \mathbb{H}_1 = \ldots = \mathbb{H}_{n+2} &= \quad , \\ \mathbb{H}_{n+3} &= \quad , \end{aligned}$$

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$$t_1(\gamma) := \ldots := t_{n+2}(\gamma) \quad = \quad \max\{1, 1/2 \cdot \gamma\},$$

$$\begin{aligned} \mathbb{H}_1 = \ldots = \mathbb{H}_{n+2} \quad &= \quad L_p(B_p^{1-1/p+1}), \\ \mathbb{H}_{n+3} \quad &= \quad , \end{aligned}$$

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$$\begin{aligned} \mathbb{H}_1 = \ldots = \mathbb{H}_{n+2} \quad &= \quad {}_0B_p^{1/2-1/(2p)+1/2}(L_p) \cap L_p(B_p^{2-1/p}), \\ \mathbb{H}_{n+3} \quad &= \quad , \end{aligned}$$

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$$t_{n+3}(\gamma) \quad := \quad \max\{2, 1 \cdot \gamma + 1, 3/2\gamma\}.$$

$$\begin{aligned} \mathbb{H}_1 = \ldots = \mathbb{H}_{n+2} &= {}_0B_p^{1-1/(2p)}(L_p) \cap L_p(B_p^{2-1/p}), \\ \mathbb{H}_{n+3} &= L_p(B_p^{1-1/p+2}), \end{aligned}$$

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$$\begin{aligned} \mathbb{H}_1 = \ldots = \mathbb{H}_{n+2} \quad &= \quad {}_0B_p^{1-1/(2p)}(L_p) \cap L_p(B_p^{2-1/p}), \\ \mathbb{H}_{n+3} \quad &= \quad {}_0B_p^{1/2-1/(2p)+3/2}(L_p) \cap {}_0H_p^1(B_p^{2-1/p}) \cap L_p(B_p^{3-1/p}), \end{aligned}$$

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$$
\begin{aligned}
\mathbb{H}_1 = \ldots = \mathbb{H}_{n+2} &= {}_0B_p^{1-1/(2p)}(L_p) \cap L_p(B_p^{2-1/p}),\\
\mathbb{H}_{n+3} &= {}_0B_p^{2-1/(2p)}(L_p) \cap {}_0H_p^1(B_p^{2-1/p}) \cap L_p(B_p^{3-1/p}),\\
\mathbb{H}_{n+4} &= {}_0B_p^{1/2-1/(2p)}(L_p) \cap L_p(B_p^{1-1/p}),
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{F}_1 = \mathbb{F}_2 &= {}_0B_p^{1/2-1/(2p)}(L_p) \cap L_p(B_p^{1-1/p}),\\
\mathbb{F}_3 &= {}_0B_p^{1-1/(2p)}(L_p) \cap L_p(B_p^{2-1/p}),\\
\mathbb{F}_4 = \ldots = \mathbb{F}_{n+4} &= {}_0B_p^{1/2-1/(2p)}(L_p) \cap L_p(B_p^{1-1/p}),
\end{aligned}
$$

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\end{aligned}$$

$$\begin{aligned}
\mathbb{F}_1 = \mathbb{F}_2 &= {}_0B_p^{1/2-1/(2p)}(L_p) \cap L_p(B_p^{1-1/p}), \\
\mathbb{F}_3 &= {}_0B_p^{1-1/(2p)}(L_p) \cap L_p(B_p^{2-1/p}), \\
\mathbb{F}_4 = \ldots = \mathbb{F}_{n+4} &= {}_0B_p^{1/2-1/(2p)}(L_p) \cap L_p(B_p^{1-1/p}),
\end{aligned}$$

So we finally obtain $\mathscr{L}_\mu(\nabla_+) \in L_{\mathrm{Isom}}\left(\prod_{k=1}^{n+4} \mathbb{H}_k, \prod_{k=1}^{n+4} \mathbb{F}_k\right)$

Other models with **gravity** or **surface viscosity** (Boussinesq-Scriven surface fluid) can be handled in essentially the same way with only slight modifications.

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- Algorithmic approach
- We don't have to calculate the inverse matrix of the system.
- We can substitute 'hard analysis' by linear algebra.
- It is easy to find spaces for maximal regularity.

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Thank you for your attention.