Mixed Order Systems and Free Boundary Value Problems

Mario Kaip

Universität Konstanz

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Contents

What is a mixed order system?

Roughly speaking this is matrix valued function $(\lambda, \xi) \in \mathbb{C} \times \mathbb{C}^n \mapsto \mathscr{L}(\lambda, \xi) \in \mathbb{C}^{m\times m}$ (also called *symbol*) which enables us to define an operator of the form

$$\begin{aligned} \mathscr{L}(\partial_t, \nabla) : \prod_{k=1}^{m} \mathbb{H}_k &\longrightarrow \prod_{k=1}^{m} \mathbb{F}_k, \\ (f_1, \dots, f_m)^T &\mapsto \mathcal{L}^{-1}\mathscr{F}^{-1}\mathscr{L}(\lambda, \xi)\mathscr{F}\mathcal{L}(f_1, \dots, f_m)^T \end{aligned}$$

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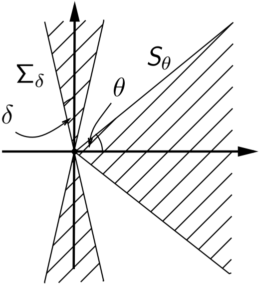
- In which cases we have $\mathscr{L}(\partial_t, \nabla) \in L_{\mathrm{Isom}}(\prod_{k=1}^m \mathbb{H}_k, \prod_{k=1}^m \mathbb{F}_k)$?

How to define $f(\partial_t, \nabla)$?

Outline: In the following we always assume $f \in H_P(S_\theta \times \Sigma_\delta^n)$ i.e

$$f : S_\theta \times \Sigma_\delta^n \longrightarrow \mathbb{C}, \qquad \theta \in (0, 2\pi), \delta \in (0, \pi/2)$$

is holomorphic and polynomially bounded.



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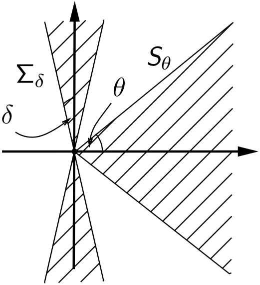

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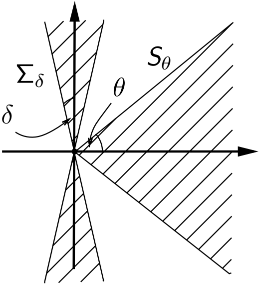

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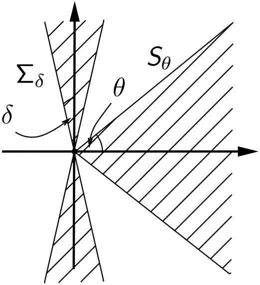
$$W^{s,r}_{\mathcal{F},\mathcal{K}} := {}_0\mathcal{F}^s(\mathbb{R}_+, \mathcal{K}^r(\mathbb{R}^n)), \quad s \geq 0, r \in \mathbb{R}$$

with

$$\mathcal{F} \in \begin{cases} \{B_{p_0,q_0}\,,\; H_{p_0}\} & , s > 0 \\ \{H_{p_0}\} & , s = 0 \end{cases}, \qquad \mathcal{K} \in \{B_{p_1,q_1}\,,\; H_{p_1}\},$$

$p_i, q_i \in (1, \infty)$.



Definition of order functions and weight functions

- A function

$$\mathcal{O}(\gamma) = \max_{\ell=0,\dots,M} \{b_\ell(\mathcal{O}) + \gamma \cdot m_\ell(\mathcal{O})\}, \quad \gamma > 0.$$

  with $m_\ell(\mathcal{O}), b_\ell(\mathcal{O}) \geq 0$ is called **order function**.

- We also define the associated **weight function** by

$$\Xi_{\mathcal{O}}(\lambda, z) := \sum_{(s,r)\in\nu(\mathcal{O})} |\lambda|^s |z|^r, \quad (\lambda, z) \in S_\theta \times \Sigma_\delta^n$$

  with
$$\nu(\mathcal{O}) := \{(m_\ell(\mathcal{O}), b_\ell(\mathcal{O})) : \ell = 0, \dots, M\} \cup\{(0,0), (\max_\ell m_\ell(\mathcal{O}), 0), (0, \max_\ell b_\ell(\mathcal{O}))\}.$$

- $\mathcal{O}$ is called an **upper (resp. lower) order function of Q** if there exist $C > 0$ and $\lambda_0 > 0$ with

$$|Q(\lambda, z)| \leq C \cdot \Xi_{\mathcal{O}}(\lambda, z), \quad (\lambda, z) \in S_\theta \times \Sigma_\delta^n, \quad |\lambda| \geq \lambda_0$$

  (resp. $\Xi_{\mathcal{O}}(\lambda, z) \leq C \cdot |Q(\lambda, z)|$).

- A symbol $Q \in H_P(S_\theta \times \Sigma_\delta^n)$ is called **N-parabolic** if there exists an order function $\mathcal{O}(Q)$ which is an upper and lower order function of Q i.e. $|Q| \approx \Xi_{\mathcal{O}(Q)}$.
  $\rightsquigarrow$ Useful characterization by non-vanishing principal parts (cf. S. Gindikin and L.R. Volevich (1992), R. Denk, J. Saal and J. Seiler (2008), K. (2009))
- **Example:** The order function $\mathcal{O}(\gamma) := \max\{1, 1/2 \cdot \gamma\}$ is a lower and upper order function of $\omega(\lambda, z) := \sqrt{\lambda - \sum_{k=1}^n z^2} \Rightarrow \omega$ is N-parabolic.

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- **Example:** Let $\mathcal{O}(\gamma) := \max\{3/2, 1 + 1\gamma, 2\gamma\}$ be upper order of Q

$$\Rightarrow \quad Q_\mu(\nabla_+^{\mathcal{W}}) \in L\left(W_{\mathcal{F},\mathcal{K}}^{s',r'+3/2} \cap W_{\mathcal{F},\mathcal{K}}^{s'+1,r'+1} \cap W_{\mathcal{F},\mathcal{K}}^{s'+2,r'}, \mathcal{W}\right)$$

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If $\mathcal{O}(\gamma) := \max\{3/2, 1 + 1\gamma, 2\gamma\}$ is also a lower order function of Q (i.e. Q is N-parabolic) then we even have

$$Q_\mu(\nabla_+^{\mathcal{W}}) \in L_{\mathrm{Isom}}\left(W_{\mathcal{F},\mathcal{K}}^{s',r'+3/2} \cap W_{\mathcal{F},\mathcal{K}}^{s'+1,r'+1} \cap W_{\mathcal{F},\mathcal{K}}^{s'+2,r'}, \mathcal{W}\right).$$

Douglis-Nirenberg/N-parabolic system

Definition

The system $\mathscr{L} \in [H_P(S_\theta \times \Sigma_\delta^n)]^{m \times m}$ is called a **mixed order system in the sense of Douglis-Nirenberg** if there are order functions $s_j := \mathcal{O}_j^{\text{row}}$ and $t_k := \mathcal{O}_k^{\text{col}}$ $(j, k = 1, \ldots, M)$ such that $s_j + t_k$ is an upper order function of $\mathscr{L}_{jk}$ for all $j, k = 1, \ldots, M$ (i.e. the upper order structure of each component splits into row and column part).

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(i) $\det \mathscr{L}$ is N-parabolic,

(ii) $[\mathcal{O}(\det \mathscr{L})](\gamma) = \sum_{j=1}^m (s_j(\gamma) + t_j(\gamma))$ for all $\gamma > 0$.

Motivation for these Definitions: Consider the N-parabolic 2×2-system $\mathscr{L} \in [H_P(S_\theta \times \Sigma_\delta^n)]^{2\times 2}$. Hence we have

$$\mathscr{L}^{-1} = \begin{pmatrix} \frac{\mathscr{L}_{22}}{\mathscr{L}_{11}\mathscr{L}_{22} - \mathscr{L}_{12}\mathscr{L}_{21}} & \frac{-\mathscr{L}_{12}}{\mathscr{L}_{11}\mathscr{L}_{22} - \mathscr{L}_{12}\mathscr{L}_{21}} \\ \frac{-\mathscr{L}_{21}}{\mathscr{L}_{11}\mathscr{L}_{22} - \mathscr{L}_{12}\mathscr{L}_{21}} & \frac{\mathscr{L}_{11}}{\mathscr{L}_{11}\mathscr{L}_{22} - \mathscr{L}_{12}\mathscr{L}_{21}} \end{pmatrix}$$

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Theorem (K., 2010)

Let $\mathscr{L} \in [H_P(S_\theta \times \Sigma_\delta^n)]^{m\times m}$ be an N-parabolic mixed order system and

$$\mathbb{H}_i := \bigcap_{\ell=0}^{M} W^{s'_\ell+m_\ell(t_i),r'_\ell+b_\ell(t_i)}_{\mathcal{F}_\ell,\mathcal{K}_\ell}, \qquad \mathbb{F}_j := \bigcap_{\ell=0}^{M} W^{s'_\ell-m_\ell(s_j),r'_\ell-b_\ell(s_j)}_{\mathcal{F}_\ell,\mathcal{K}_\ell}$$

with fixed constants $s'_\ell, r'_\ell \geq 0$. If certain "compatibility embeddings" hold then there exists $\mu_0 > 0$ such that for all $\mu \geq \mu_0$

$$\mathscr{L}_\mu(\nabla_+) \in L_{Isom}(\mathbb{H},\mathbb{F}), \qquad (\mathscr{L}_\mu(\nabla_+))^{-1} = \mathscr{L}_\mu^{-1}(\nabla_+)$$

with $\mathbb{H} := \prod_{i=1}^m \mathbb{H}_i$ and $\mathbb{F} := \prod_{i=1}^m \mathbb{F}_i$.

Applications

We can apply these results to the following problems
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Two-phase Navier Stokes equation with surface tension

We consider the following linearized problem for $u = (v, w)$, π, and h:

$$\begin{cases} \rho\partial_t u - \mu\Delta u + \nabla\pi = 0 & \text{in } \mathbb{R}_+ \times \dot{\mathbb{R}}^{n+1} \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \dot{\mathbb{R}}^{n+1} \\ -[\![\mu\partial_y v]\!] - [\![\mu\nabla_x w]\!] = g_v & \text{on } \mathbb{R}_+ \times \mathbb{R}^n \\ -2[\![\mu\partial_y w]\!] + [\![\pi]\!] - \sigma\Delta h = g_w & \text{on } \mathbb{R}_+ \times \mathbb{R}^n \\ [\![u]\!] = 0 & \text{on } \mathbb{R}_+ \times \mathbb{R}^n \\ \partial_t h - \gamma_0 w = g_h & \text{on } \mathbb{R}_+ \times \mathbb{R}^n \\ u(t=0) = 0 & \text{in } \dot{\mathbb{R}}^{n+1} \\ h(t=0) = 0 & \text{in } \mathbb{R}^n \end{cases}$$

with $(u, \pi, \rho, \mu) = (u_1, \pi_1, \rho_1, \mu_1)\chi_{\mathbb{R}^{n+1}_-} + (u_2, \pi_2, \rho_2, \mu_2)\chi_{\mathbb{R}^{n+1}_+}$, $\sigma > 0$,
$[\![\phi]\!] := \gamma_0\phi_{\mathbb{R}^{n+1}_+} - \gamma_0\phi_{\mathbb{R}^{n+1}_-}$.

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- Formal Laplace respectively Fourier transform yield ODEs for $(\hat{u}, \hat{\pi})$.
- Solve ODEs with Green's functions.
- Determine free parameters by boundary conditions.
- Solve $[\![\hat{w}]\!] = 0$ to eliminate pressure trace $\hat{p}_2$.

Two-phase Navier Stokes equation with surface tension

For the unknowns $\hat{\Phi}_v^{(j)}$, $\hat{\Phi}_w^{(j)}$, $[\![\hat{\pi}]\!]$, and $\hat{h}$ we obtain the system

$$\underbrace{\begin{pmatrix} i\xi^T & -\omega_2 & 0 & 0 & 0 \\ i\xi^T & 0 & \omega_1 & 0 & 0 \\ 0 & -\mu_2\omega_2\gamma_2^+\Omega_+^{-1} & -\mu_1\omega_1\gamma_1^+\Omega_+^{-1} & \lambda & -|\xi|\Omega_+^{-1} \\ 0 & 2\mu_2\omega_2 & 2\mu_1\omega_1 & \sigma|\xi|^2 & 1 \\ \Omega\mathrm{id}_n & -i\xi(\mu_2+\kappa) & i\xi(\mu_1+\kappa) & 0 & i\xi(\mu_2\gamma_2^- - \mu_1\gamma_1^-)\Omega_+^{-1} \end{pmatrix}}_{=:\mathscr{L}} \begin{pmatrix} \hat{\Phi}_v^{(2)} \\ \hat{\Phi}_w^{(2)} \\ \hat{\Phi}_w^{(1)} \\ \hat{h} \\ [\![\hat{\pi}]\!] \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \hat{g}_h \\ \hat{g}_w \\ \hat{g}_v \end{pmatrix}, \quad |\xi|\hat{p}_2 = -\delta\hat{\Phi}_w^{(2)} + \delta\hat{\Phi}_w^{(1)} + \frac{1}{\mu_1}\cdot\frac{\delta|\xi|}{\omega_1\gamma_1^+}[\![\hat{\pi}]\!]$$

with $\omega_j(\lambda,\xi) := \mu_j^{-1/2}(\rho_j\lambda + \mu_j|\xi|^2)^{1/2}$, $\gamma_j^\pm := \omega_j \pm |\xi|$, $\delta := \frac{\mu_1\mu_2\omega_1\omega_2\gamma_1^+\gamma_2^+}{\Omega_+}$
$\Omega_+ := \mu_1\omega_1\gamma_1^+ + \mu_2\omega_2\gamma_2^+$, $\kappa := \frac{\mu_1\mu_2}{|\xi|}\cdot(\omega_1\gamma_1^+\gamma_2^- + \omega_2\gamma_2^+\gamma_1^-)\Omega_+^{-1}$

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As determinant we obtain $|\det \mathscr{L}| = |\omega_1\omega_2/\Omega_+| \cdot |\Omega^{n-1}| \cdot |P|$ with

$$\begin{aligned} P(\lambda,\xi) \quad := \quad & (\mu_1\omega_1^2 + \mu_2\omega_2^2)(\mu_1\omega_1 + \mu_2\omega_2)\lambda \\ & + \left[(\mu_1\omega_1 + \mu_2\omega_2)^2 + \mu_1\mu_2(\omega_1 + \omega_2)^2\right] \lambda|\xi| \\ & + \left[3(\mu_2^2\omega_2 + \mu_1^2\omega_1) - \mu_1\mu_2(\omega_1 + \omega_2)\right] \lambda|\xi|^2 \\ & -(\mu_1 - \mu_2)^2\lambda|\xi|^3 + \sigma(\mu_1\omega_1 + \mu_2\omega_2)|\xi|^3 \\ & +\bar{\mu}\sigma|\xi|^4 \end{aligned}$$

$\bar{\mu} := \mu_1 + \mu_2$, $\bar{\rho} := \rho_1 + \rho_2$. One can easily prove with the characterization of non-vanishing principal parts that $\det \mathscr{L}$ is N-parabolic with order function

$$[\mathcal{O}(\det \mathscr{L})](\gamma) = \max\{n+3, \gamma+n+2, [n+4]/2\gamma\}, \quad \gamma > 0.$$

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To obtain a Douglis-Nirenberg system we define

$$
\begin{aligned}
t_1(\gamma) := \ldots := t_{n+2}(\gamma) &:= \max\{1, \gamma/2\},\\
t_{n+3}(\gamma) &:= \max\{2, \gamma+1, 3/2\gamma\},\\
t_{n+4}(\gamma) &:= 0,\\
s_1(\gamma) := s_2(\gamma) &:= 0,\\
s_3(\gamma) &:= -\max\{1, \gamma/2\},\\
s_4(\gamma) := \ldots := s_{n+4}(\gamma) &:= 0.
\end{aligned}
$$

It is easy to show that $s_j + t_k$ is an upper order function of $\mathscr{L}_{j,k}$ and $\mathcal{O}(\det \mathscr{L}) = \sum_{j=1}^{n}(s_j + t_j)$.

$\Rightarrow \mathscr{L}$ is an N-parabolic mixed order system

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So we can construct the spaces of maximal regularity with $(r_0', s_0') := (r_1', s_1') := (1 - 1/p, 0)$, $(r_2', s_2') := (0, 1/2 - 1/(2p))$, and $(\mathcal{F}_0, \mathcal{K}_0) := (\mathcal{F}_1, \mathcal{K}_1) = (H_p, B_p)$, $(\mathcal{F}_2, \mathcal{K}_2) = (B_p, H_p)$

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$$\mathbb{H}_1 = \ldots = \mathbb{H}_{n+2} = {}_0B_p^{1-1/(2p)}(\mathbb{R}_+, L_p(\mathbb{R}^n)) \cap L_p(\mathbb{R}_+, B_p^{2-1/p}(\mathbb{R}^n)),$$

$$\begin{aligned} \mathbb{H}_{n+3} &= {}_0B_p^{2-1/(2p)}(\mathbb{R}_+, L_p(\mathbb{R}^n)) \cap {}_0H_p^1(\mathbb{R}_+, B_p^{2-1/p}(\mathbb{R}^n)) \cap L_p(\mathbb{R}_+, B_p^{3-1/p}(\mathbb{R}^n)), \\ \mathbb{H}_{n+4} &= {}_0B_p^{1/2-1/(2p)}(\mathbb{R}_+, L_p(\mathbb{R}^n)) \cap L_p(\mathbb{R}_+, B_p^{1-1/p}(\mathbb{R}^n)), \end{aligned}$$

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$$\mathbb{H}_1 = \ldots = \mathbb{H}_{n+2} = {}_0B_p^{1-1/(2p)}(\mathbb{R}_+, L_p(\mathbb{R}^n)) \cap L_p(\mathbb{R}_+, B_p^{2-1/p}(\mathbb{R}^n)),$$

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\mathbb{F}_3 &= {}_0B_p^{1-1/(2p)}(\mathbb{R}_+, L_p(\mathbb{R}^n)) \cap L_p(\mathbb{R}_+, B_p^{2-1/p}(\mathbb{R}^n)), \\
\mathbb{F}_4 = \ldots = \mathbb{F}_{n+4} &= {}_0B_p^{1/2-1/(2p)}(\mathbb{R}_+, L_p(\mathbb{R}^n)) \cap L_p(\mathbb{R}_+, B_p^{1-1/p}(\mathbb{R}^n)),
\end{aligned}$$

So we finally obtain $\mathscr{L}_\mu(\nabla_+) \in L_{\text{Isom}}\left(\prod_{k=1}^{n+4} \mathbb{H}_k, \prod_{k=1}^{n+4} \mathbb{F}_k\right)$.

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- Algorithmic approach
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Thank you for your attention.