The Dynamic Convex Valuation Related To The Price Process In A Market With General Jumps *

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Abstract. We consider an incomplete market with general jumps in the given price process S of a risky asset. We define the S*-related dynamic convex valuation* (S-related DCV) which is time-consistent. We discuss the representation for a given S-related DCV C in terms of a 'penalty functional' α and give some characteristics of α, which are the sufficient conditions for a given C to be an S-related DCV. Finally, we give two special forms of α satisfying those conditions to describe the dynamics of the corresponding S-related DCV by a backward semimartingale equation.

Key words: backward semimartingale equation (BSE); dynamic convex valuation (DCV); dynamic convex risk measure; time-consistency

Mathematics Subject Classification (2000): 90A09, 60H30, 60G44

1 Introduction

One of the important problems in mathematical finance is the valuation of contingent claims in incomplete financial markets. Recently, many researchers have studied the utility indifference valuation method, see Klöppel and Schweizer(2007)[19], Mania and Schweizer (2005)[24], Rouge and El Karoui (2000)[26], and the references in there. In this paper, we continue our research on valuations in Xiong and Kohlmann (2008)[27] where a special utility indifference approach is described. In this paper we avoid the use of a utility function and propose a more direct model for valuations.

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We consider a financial market similar to the one in Kohlmann and Xiong(2007)[20]. In the market, there is a risky asset whose discounted price process $S = (S_t; t \in [0,T])$ is a semimartingale with bounded jumps. We assume that all the purely discontinuous local martingales are driven by an integer-valued random measure $\mu(\omega; du, dy)$ and the continuous martingale part of S is driven a continuous local martingale M. Under some assumptions, the market is an incomplete market and we consider the S-related dynamic convex valuation of a contingent claim B.

Mania and Schweizer (2005)[24] studied the dynamics of the exponential utility indifference value process $C(B;\alpha)$ for a contingent claim B in a semimartingale model with a general continuous filtration. One can show even in our jump model that for fixed $t \in [0,T]$ the exponential utility indifference valuation $C_t(B;\alpha)$ satisfies the desirable properties of a valuation

(A) **Monotonicity**: $C_t(B_1;\alpha) \leq C_t(B_2;\alpha)$ for all $B_1, B_2 \in L^\infty(\mathscr{F}_T)$ with $B_1 \leq B_2$;

(B) $\mathscr{F}_t$**-convexity**: $C_t(\lambda B_1 + (1-\lambda)B_2;\alpha) \leq \lambda C_t(B_1;\alpha) + (1-\lambda)C_t(B_2;\alpha)$ for all $\lambda \in L^0(\mathscr{F}_t)$ with values in $[0,1]$ and $B_1, B_2 \in L^\infty(\mathscr{F}_T)$;

(C) $\mathscr{F}_t$**-translation invariance**: $C_t(B + a_t;\alpha) = C_t(B;\alpha) + a_t$ for all $B \in L^\infty(\mathscr{F}_T)$ and $a_t \in L^\infty(\mathscr{F}_t)$;

(D) $\mathscr{F}_t$**-regularity**: $C_t(B_1 I_A + B_2 I_{A^c};\alpha) = C_t(B_1;\alpha)I_A + C_t(B_2;\alpha)I_{A^c}$ for all $B_1, B_2 \in L^\infty(\mathscr{F}_T)$ and $A \in \mathscr{F}_t$;

(E) **Time consistency**: for all $B_1, B_2 \in L^\infty(\mathscr{F}_T)$ and all $s \in [0,t]$

$$C_t(B_1;\alpha) = C_t(B_2;\alpha) \quad \text{implies that} \quad C_s(B_1;\alpha) = C_s(B_2;\alpha) \; .$$

Such properties are also found in the concept of coherent measures of risk (see Artzner et al.(1999)[1] or Delbaen(2002)[8]), in the concept of convex measures of risk (see Föllmer and Schied(2002)[12]), and in the concept of dynamic monetary concave utility functionals (DMCUF, see Klöppel and Schweizer(2007)[19]). Here -for the first time, to the best of our knowledge- we consider another interesting property

(F) S**-related property**: for all $\pi \in Adm$ and $x \in L^\infty(\mathscr{F}_0)$,

$$C_t(X_T^{x,\pi}) = X_t^{x,\pi}, \quad \text{a.s.},$$

where $X^{x,\pi} = x + \int \pi_u dS_u$ is the corresponding wealth process related to (x,π),

and a valuation $C = \{C(B) = (C_t(B))_{t \in [o,T]}; B \in L^\infty(\mathscr{F}_T)\}$ satisfying (A)$\sim$(F) will be called the *S-related dynamic convex valuation* (S-related DCV). In this way, we incorporate into the valuation the fact that the value at time $t \leq T$ of an attainable wealth at time T should be the wealth itself at time t. Similar to Theorem 3.13 of Klöppel and Schweizer(2007)[19], we obtain a representation for the S-related DCV as the following

Theorem 1.1. *If $C = \{C(B) = (C_t(B))_{t\in[0,T]}, B \in L^\infty(\mathscr{F}_T)\}$ is a dynamic convex valuation related to S, which is continuous from above at time t, then there exists a "penalty functional" α such that for all $B \in L^\infty(\mathscr{F}_T)$*

$$C_t(B) = \operatorname*{ess\,sup}_{Z\in\mathbb{Z}_e} \left\{ E\left[Z_{t,T}B \middle| \mathscr{F}_t\right] - \alpha_t(Z_{t,T}) \right\},$$

where $\mathbb{Z}_e$ is the family of the density processes of all equivalent martingale measure. *Here, for each $Z \in \mathbb{Z}_e$, $Z_{t,T} := \frac{Z_T}{Z_t}$ and*

$$\alpha_t(Z_{t,T}) := \operatorname*{ess\,sup}_{B\in L^\infty(\mathscr{F}_T)} \left\{ E\left[Z_{t,T}B \middle| \mathscr{F}_t\right] - C_t(B) \right\}$$

is the convex conjugate of C_t.

Different from Klöppel and Schweizer(2007)[19], we here consider the family of the density processes of all equivalent martingale measures $\mathbb{Z}_e$ in order to be able to treat the additionally introduced S-related property. Furthermore, we discuss the property of α and try to give sufficient conditions for a given α to be the penalty functional of an S-related DCV. Then we give two special versions of α satisfying those conditions. This then allows to describe the dynamics of the corresponding S-related DCV by a backward semimartingale equation.

The paper is structured as follows. An incomplete market model with general jumps is established in section 2. We give the form of the density process of the equivalent martingale measure and give the definition of the S-related DCV. In section 3, we state a representation theorem for the S-related DCV by a 'penalty functional' α, and then we discuss some properties of α which are the sufficient for α to be the penalty functional of a S-related DCV. In section 4, we give two versions of α satisfying the conditions derived in section 3 and describe the dynamics of the corresponding S-related DCV by a backward semimartingale equation.

2 The preliminaries

In this paper, we consider the dynamic convex valuation in a market similar to the one in Kohlmann and Xiong(2007)[20]. We begin with a finite time horizon $T > 0$ and a complete probability space $(\Omega, \mathscr{F}, P)$. Let $\mathbb{F} = \{\mathscr{F}_t, 0 \le t \le T\}$ be a filtration satisfying the usual conditions with $\mathscr{F}_T = \mathscr{F}$. We assume that there exists a risky asset in the market, whose discounted price $S = (S_t)_{0\le t\le T}$ is described by a special semimmartingale with bounded jumps. To describe the dynamics of S, we need the following notations:

- $A = (A_t)_{0\le t\le T}$ is an $\mathbb{F}$-adapted continuous increasing process with $A_0 = 0$ and $EA_T < \infty$;
- $M = (M_t)_{0\le t\le T}$ is a continuous local martingale with $\langle M, M\rangle_t = A_t$;
- $\mu = \{\mu(\omega; dt, dx) : \omega \in \Omega\}$ is an integer-valued random measure on $\mathbb{R}_+ \times \mathbb{R}$ with compensator $\nu(\omega; dt, dx) = dA_t(\omega)K(\omega, t; dx)$, where $K(\omega, t; dx)$ is a kernel from $(\Omega \times \mathbb{R}_+, \mathscr{P})$ into $(\mathbb{R}, \mathscr{B})$. Let $\widetilde{N}(\omega; dt, dx) := \mu(\omega; dt, dx) - \nu(\omega; dt, dx)$.

Throughout the paper, we assume that all the jump martingales in the market are driven by the random measure μ, and that the following representation holds:

Assumption 2.1. *Any purely discontinuous local $(P, \mathbb{F})$-martingale m can be represented in the following form*

$$m_t = m_0 + \int_0^t \int_{\mathbb{R}} \psi(s,x)\widetilde{N}(ds,dx).$$

Remark 2.2.

1. *Note that here we do not assume any representation property with respect to the continuous local martingale. Also the market model discussed in this paper is more general than the market considered in Kohlmann and Xiong (2007,a)[20].*
2. *As A is a continuous process, $a_t(\omega) := \nu(\omega, \{t\}\times\mathbb{R}) = 0$. Thus for all $\widetilde{\mathscr{P}} := \mathscr{P}\otimes\mathscr{B}$-measurable functions $W(\omega,t,x)$, let*

$$\widehat{W}_t(\omega) := \begin{cases} \int_{\mathbb{R}} W(\omega,t,x)\nu(\omega;\{t\}\times dx) & \text{if } \int_{\mathbb{R}} |W(\omega,t,x)|\nu(\omega;\{t\}\times dx) < \infty \\ +\infty & \text{otherwise ,} \end{cases}$$

   then $\widehat{W} = 0$. Thus Assumption 2.1 implies that the filtration $\mathbb{F}$ is quasi-left continuous.
3. *It is easy to see that there exists an $\mathbb{F}$-optional process $\beta = (\beta_t)_{0\le t\le T}$ and a sequence of stopping times $(\hat{\tau}_n)$ such that for all positive $\widetilde{\mathscr{P}}$-mesurable function $W(\omega,t,x)$,*

$$W * \mu_t := \int_0^t \int_{\mathbb{R}} W(\omega,u,x)\mu(\omega;du,dx) = \sum_{(n)} W(\hat{\tau}_n, \beta_{\hat{\tau}_n}) I_{\{\hat{\tau}_n \le t\}}. \qquad \square$$

We now describe the market model. We assume that the dynamics of the discounted price process S are given by

$$dS_t = S_0 + \int_0^t \gamma_u dA_u + \int_0^t \sigma_1(u)dM_u + \int_0^t \int_{\mathbb{R}} \sigma_2(u,y)\widetilde{N}(du,dy),$$

where γ and σ_1 are two predictable processes with $\int_0^T \sigma_1(s)^2 dA_s < \infty$ a.s. and $\sigma_2(t,y)$ is a bounded $\widetilde{\mathscr{P}}$-measurable function. We assume that there exist two constants k and K such that

$$\int_0^T \gamma_u^2 dA_u \le K < \infty \text{ and } \sigma_1(s) \ge k > 0. \tag{2.1}$$

Just as in Kohlmann and Xiong(2007,a)[20], let $\delta_t = \dfrac{\gamma_t}{\sigma_1(t)}$ to introduce a new measure Q^0 by

$$\frac{dQ^0}{dP}\Big|_{\mathscr{F}} := Z_T^0,$$

where $Z_t^0 = \mathscr{E}(-\int \delta_u dM_u)_t$. From $\int_0^T \delta_u^2 dA_u \le \dfrac{1}{k^2}\int_0^T \gamma_u^2 dA_u \le \dfrac{K}{k^2} < \infty$, Z^0 is a continuous uniformly integrable martingale so that $Z^0 S$ is a local martingale. Thus Q^0 is an equivalent martingale measure. If we let $\widetilde{M}_t := M_t + \int_0^t \delta_u dA_u$, it is easy to see that $\widetilde{M}$ is a continuous local Q^0-martingale. Hence, under Q^0, the discounted price process S can be rewritten as

$$S_t = S_0 + \int_0^t \sigma_1(u) d\widetilde{M}_u + \int_0^t \int_{\mathbb{R}} \sigma_2(u,y) \widetilde{N}(du,dy).$$

Remark 2.3. *(1) It follows from Proposition 2.3 of Kohlmann and Xiong (2007)[21] that Q^0 satisfies the reverse Hölder inequality $R_{LLOGL}(P)$, i.e., there exists a constant C such that for any stopping time τ*

$$E\left\{ Z^0_{\tau,T} \log Z^0_{\tau,T} \middle| \mathscr{F}_\tau \right\} \le C < \infty\ , \tag{2.2}$$

where $Z^0_{\tau,T} = \frac{Z^0_T}{Z^0_\tau}$. Furthermore, for any $B \in L^\infty(\mathscr{F}_T)$, we can define a measure P_B as in [9]:

$$\frac{dP_B}{dP}|_{\mathscr{F}_T} := \frac{e^{\alpha B}}{E e^{\alpha B}}.$$

So the density process of Q^0 with respect to P_B is given by

$$Z^{0,B}_t = E_{P_B}\left\{ Z^0_T \frac{E(e^{\alpha B})}{e^{\alpha B}} \middle| \mathscr{F}_t \right\} = E(e^{\alpha B}) \frac{Z^0_t}{E\left\{ e^{\alpha B} \middle| \mathscr{F}_t \right\}}\ ,$$

and since $e^{\alpha B}$ is bounded, one sees that Q^0 satisfies the reverse Hölder inequality $R_{LLOGL}(P_B)$, i.e.,

$$E_{P_B}\left\{ Z^{0,B}_{\tau,T} \log Z^{0,B}_{\tau,T} \middle| \mathscr{F}_\tau \right\} \le C < \infty\ , \quad a.s.$$

holds for some constant C and all stopping time τ, where $Z^{0,B}_{\tau,T} = \frac{Z^{0,B}_T}{Z^{0,B}_\tau}$.

(2) Note that the minimal entropy martingale measure, *denoted by Q^E, exists and is equivalent to P according to Frittelli(2000)[13].* □

Let

$$\mathbb{Z} = \left\{ Z \,\middle|\, \begin{array}{l} Z \text{ is a nonnegative uniformly integrable martingale with} \\ \text{RCLL paths such that } ZS \text{ is a local martingale} \end{array} \right\}.$$

For any $Z \in \mathbb{Z}$, let $\tau = \inf\{t : Z_t = 0\}$. From He, Wang and Yan(1992) τ is a stopping time so that for all $\omega \in \{\tau < \infty\}$

$$Z_t(\omega) = 0, \quad t \ge \tau(\omega),$$

and for all $\omega \in \{\tau > 0\}$ and $t < \tau(\omega)$,

$$Z_t(\omega) \neq 0 \text{ and } Z_{t-}(\omega) \neq 0\ .$$

Let $\mathbb{Z}_e = \left\{ Z \in \mathbb{Z}; Z_t > 0, a.s., \text{ for all } t \in [0,T] \right\}$, since $Z^0 \in \mathbb{Z}_e$, we have that $\mathbb{Z}_e \neq \emptyset$. For all $Z \in \mathbb{Z}_e$, define $Z_{t,T} := \frac{Z_T}{Z_t}$. We also introduce $\tilde{Z}_t = \dfrac{Z_t}{Z^0_t} = E_{Q^0}\left(\frac{Z_T}{Z^0_T} | \mathscr{F}_t \right)$ and $\tilde{\mathbb{Z}}_e = \left\{ \tilde{Z} = \frac{Z}{Z^0}; Z \in \mathbb{Z}_e \right\}$. With these notations we can state the following lemma

Lemma 2.4. *For any $Z \in \mathbb{Z}_e$, there exists a $\widetilde{\mathscr{P}}$-measurable function $l(u,y)$ with $l(u,y) > 1$ and a continuous local martingale L strongly orthogonal to M with $L_0 = 0$ such that*

$$Z_t = Z_0 \mathcal{E}\Big\{ - \int_0^{\cdot} \{\delta_u + \tilde{l}_u\} dM_u + \int_0^{\cdot} \int_{\mathbb{R}} l(u,y) \tilde{N}(du,dy) + L \Big\}_t$$

where $\tilde{l}_u := \int_{\mathbb{R}} \frac{\sigma_2(u,y)}{\sigma_1(u)} l(u,y) K(u;dy)$.

Proof. Since $Z \in \mathbb{Z}_e$, Z can be represented as

$$Z_t = Z_0 + \int_0^t Z_{u-} dm_u$$

where m is a local martingale with $m_0 = 0$. Since M is a continuous local martingale, there exists a predictable process ζ and a continuous local martingale L such that m^c, the continuous martingale part of m, can be represented as

$$m_t = \int_0^t \zeta_u dM_u + L_u.$$

Furthermore, by Assumption 2.1, m^d, the jump martingale part of m, can be written as

$$m_t^d = \int_0^t \int_{\mathbb{R}} l(u,y) \tilde{N}(du,dy),$$

thus Z is the solution of the following stochastic differential equation

$$Z_t = Z_0 + \int_0^t Z_{u-} \zeta_u dM_u + \int_0^t \int_{\mathbb{R}} Z_{u-} l(u,y) \tilde{N}(du,dy) + \int_0^t Z_{u-} dL_u \ .$$

By making use of Itô's formula, we derive

$$\begin{aligned} Z_t S_t &= Z_0 S_0 + Z_- \cdot S_t + S_- \cdot Z_t + [Z,S]_t \\ &= Z_0 S_0 + \int_0^t Z_{u-} \Big\{ \gamma_u + \sigma_1(u) \zeta_u + \sigma_1(u) \tilde{l}_u \Big\} dA_u \\ &\quad + \int_0^t Z_{u-} \{\sigma_1(u) + S_{u-} \zeta_u\} dM_u + \int_0^t Z_{u-} S_{u-} dL_u \\ &\quad + \int_0^t \int_{\mathbb{R}} Z_{u-} \Big\{ \sigma_2(u,y) + S_{u-} l(u,y) + \sigma_2(u,y) l(u,y) \Big\} \widetilde{N}(du,dy) \ . \end{aligned}$$

Since ZS is a local martingale by the definition of $\mathbb{Z}_e$, we finally get

$$\zeta_u = -\delta_u - \tilde{l}_u, \qquad dA_u \times dP \text{ -a.s.},$$

thus $Z_t = Z_0 \mathcal{E}\big\{ - \int_0^{\cdot} \{\delta_u + \tilde{l}_u\} dM_u + \int_0^{\cdot} \int_{\mathbb{R}} l(u,y) \tilde{N}(du,dy) + L \big\}_t$. □

Corollary 2.5. *For any $\tilde{Z} \in \tilde{\mathbb{Z}}_e$, there exists a $\widetilde{\mathscr{P}}$-measurable function $l(u,y)$ with $l(u,y) > 1$ and a continuous local Q^0-martingale L strongly Q^0-orthogonal to $\widetilde{M}$ (i.e., $L\widetilde{M}$ is a continuous local Q^0-martingale) with $L_0 = 0$ such that*

$$\tilde{Z}_t = Z_0 \mathcal{E}\big\{ -\int_0^\cdot \tilde{l}_u d\widetilde{M}_u + \int_0^\cdot \int_{\mathbb{R}} l(u,y)\tilde{N}(du,dy) + L\big\}_t$$

where $\tilde{l}_u := \displaystyle\int_{\mathbb{R}} \frac{\sigma_2(u,y)}{\sigma_1(u)} l(u,y)K(u;dy)$.

We consider the following family of the admissible strategies

$$Adm = \left\{ \pi \;\middle|\; \begin{array}{c} \pi = (\pi_u)_{u\in[0,T]} \text{ is a } S\text{-integrable predictable process} \\ \text{such that } \sup_{t\in[0,T]} |\int_0^t \pi_u dS_u| \in L^\infty(\mathscr{F}_T) \end{array} \right\},$$

and for $\pi \in Adm$ and $x \in L^\infty(\mathscr{F}_0)$, let $X_t^{x,\pi} := x + \displaystyle\int_0^t \pi_u dS_u$ be the corresponding wealth process. We now can give the definition of the dynamic convex valuation related to the price process S:

Definition. An S**-related dynamic convex valuation (**S**-related DCV)** is defined to be the family

$$\Big\{ C(B) := \big\{C_t(B)\big\}_{t\in[0,T]} \; ; \quad B \in L^\infty(\mathscr{F}_T) \Big\}$$

such that

(1) for given $B \in L^\infty(\mathscr{F}_T)$, $C(B) := \{C_t(B)\}_{t\in[0,T]}$ is a bounded RCLL semimartingale;

(2) for every $t \in [0,T]$, $C_t(B)$ satisfies the following properties:

(A) **Monotonicity**: $C_t(B_1) \le C_t(B_2)$ for all $B_1, B_2 \in L^\infty(\mathscr{F}_T)$ with $B_1 \le B_2$;

(B) $\mathscr{F}_t$**-convexity**: $C_t(\lambda B_1 + (1-\lambda)B_2) \le \lambda C_t(B_1) + (1-\lambda)C_t(B_2)$ for all $\lambda \in L^0(\mathscr{F}_t)$ with values in $[0,1]$ and $B_1, B_2 \in L^\infty(\mathscr{F}_T)$;

(C) $\mathscr{F}_t$**-translation invariance**: $C_t(B + a_t) = C_t(B) + a_t$ for all $B \in L^\infty(\mathscr{F}_T)$ and $a_t \in L^\infty(\mathscr{F}_t)$;

(D) $\mathscr{F}_t$**-regularity**: $C_t(B_1 I_A + B_2 I_{A^c}) = C_t(B_1)I_A + C_t(B_2)I_{A^c}$ for all $B_1, B_2 \in L^\infty(\mathscr{F}_T)$ and $A \in \mathscr{F}_t$;

(E) **Time consistency**: for all $B_1, B_2 \in L^\infty(\mathscr{F}_T)$ and all $s \in [0,t]$

$$C_t(B_1) = C_t(B_2) \quad \text{implies that} \quad C_s(B_1) = C_s(B_2)\;;$$

(F) S**-related property**: for all $\pi \in Adm$ and $x \in L^\infty(\mathscr{F}_0)$,

$$C_t(X_T^{x,\pi}) = X_t^{x,\pi}, \quad \text{a.s.},$$

where $X^{x,\pi} = x + \int \pi_u dS_u$ is the wealth process related to (x,π).

Example. For a fixed positive constant $\alpha > 0$ and $B \in L^\infty(\mathscr{F}_T)$, introduce

$$C_t(B;\alpha) := \frac{1}{\alpha} \log \operatorname*{ess\,inf}_{\pi \in Adm} E_{Q^E}\left(e^{\alpha(B - X^{x,\pi}_{t,T})} \big| \mathscr{F}_t \right),$$

where $X^{x,\pi}_{t,T} = X^{x,\pi}_T - X^{x,\pi}_t = \int_t^T \pi_u dS_u$. It easily follows from Mania and Schweizer(2005)[24] that $C(B;\alpha)$ is an S-related DCV.

3 The representation theorem

We now derive the representation theorem for the S-related DCV of a contingent claim. We begin with the static case.

Definition. A mapping $\rho : L^\infty(\mathscr{F}_T) \longrightarrow \mathbb{R}$ is called a **static S-related convex valuation** if it satisfies

(i) **Monotonicity**: $\rho(B_1) \le \rho(B_2)$ for all $B_1, B_2 \in L^\infty(\mathscr{F}_T)$ with $B_1 \le B_2$;

(ii) **Convexity**: $\rho(\lambda B_1 + (1-\lambda)B_2) \le \lambda\rho(B_1) + (1-\lambda)\rho(B_2)$ for all constant λ in $[0,1]$ and $B_1, B_2 \in L^\infty(\mathscr{F}_T)$;

(iii) **Translation invariance**: $\rho(B + a) = \rho(B) + a$ for all $B \in L^\infty(\mathscr{F}_T)$ and constant a;

(iv) **S-related property**: $\rho(X^{0,\pi}_T) = 0$ for all $\pi \in Adm$.

As in [12], we let $\mathcal{A}_\rho = \{B \in L^\infty(\mathscr{F}_T) \big| \rho(B) \le 0\}$, so that $\mathcal{A}_\rho$ is a convex set.

Lemma 3.1. *Assume that $\rho : L^\infty(\mathscr{F}_T) \longrightarrow \mathbb{R}$ is a static S-related convex valuation and $\mathcal{A}_\rho$ is $\sigma(L^\infty(Q^0), L^1(Q^0))$-closed, then there exists a "penalty functional" $\hat\alpha$ such that for every $B \in L^\infty(\mathscr{F}_T)$*

$$\rho(B) = \sup_{Z \in \mathbb{Z}} \{E[Z_{0,T}B] - \hat\alpha(Z)\},$$

where $\hat\alpha$ is the convex conjugate of ρ, i.e.,

$$\hat\alpha(Z) := \sup_{B \in L^\infty(\mathscr{F}_T)} \{E[Z_{0,T}B] - \rho(B)\} \ .$$

Proof. The idea is the same as in the proof of Theorem 5 of [12]. It is easy to see that

$$\hat\alpha(Z) = \sup_{B \in \mathcal{A}_\rho} \{E[Z_{0,T}B]\} \ ,$$

and for any $B \in L^\infty(\mathscr{F}_T)$ and $Z \in \mathbb{Z}$,

$$E[Z_{0,T}B] - \hat\alpha(Z) \le E[Z_{0,T}B] - \{E[Z_{0,T}B] - \rho(B)\} = \rho(B).$$

Thus we must show that for any $m \in \mathbb{R}$ such that

$$m > \sup_{Z \in \mathbb{Z}} \{E[Z_{0,T}B] - \hat\alpha(Z)\},$$

we have $m \geq \rho(B)$, i.e., $B - m \in \mathcal{A}_\rho$. Suppose that, on the contrary, $B - m \notin \mathcal{A}_\rho$. Since $\mathcal{A}_\rho$ is a $\sigma(L^\infty(Q^0), L^1(Q^0))$-closed convex set one can find a linear functional ℓ such that

$$\beta := \sup_{B \in \mathcal{A}_\rho} \ell(B) < \ell(B - m) =: \gamma < \infty.$$

We claim that

1. ℓ is a positive linear functional;
2. for any $\pi \in Adm$, $\ell(X_T^{0,\pi}) = 0$.

In fact, for any $B \in L^\infty(\mathscr{F}_T)$ with $B \geq 0$, one can see that for all $\lambda \geq 0$, $\rho(-\lambda B) \leq \rho(0) = 0$, thus $-\lambda B \in \mathcal{A}_\rho$ and

$$\gamma > \ell(-\lambda B) = -\lambda \ell(B),$$

it follows by taking $\lambda \uparrow \infty$ that $\ell(B) \geq 0$. Further more, if there exists a $\pi^* \in Adm$ such that $\ell(X_T^{0,\pi^*}) \neq 0$, let $c = \frac{\gamma}{\ell(X_T^{0,\pi^*})}$, then $\ell(cX_T^{0,\pi^*}) = \gamma$, which is a contradiction to $cX_T^{0,\pi^*} \in \mathcal{A}_\rho$. Therefore, there exists a $\xi \in L^1(Q^0, \mathscr{F}_T)$ with $\xi \geq 0$ such that

$$\ell(B) = E_{Q^0}[B\xi]$$

holds for all $B \in L^\infty(\mathscr{F}_T)$. Since $E[|Z_T^0 \xi|] = E_{Q^0}(\xi) < \infty$, let

$$\hat{Z}_t = E\left(Z_T^0 \xi \middle| \mathscr{F}_t\right),$$

one gets that $\hat{Z}$ is a nonnegative uniformly integrable martingale which has an RCLL version, still denoted by $\hat{Z}$. Since S is a local bounded semimartingale, there exists a sequence of stopping times (τ_n) such that S^{τ_n} is bounded for each n. For any stopping time $\tau \leq T$

$$E[\hat{Z}_{\tau\wedge\tau_n} S_{\tau\wedge\tau_n}] = E[\hat{Z}_T S_{\tau\wedge\tau_n}] = \ell(\int_0^T I_{[\![0,\tau\wedge\tau_n]\!]}(u) dS_u) = 0,$$

thus $\hat{Z}^{\tau_n} S^{\tau_n}$ is a true martingale for each n and $\hat{Z}S$ is a local martingale under P. Thus $\hat{Z} \in \mathbb{Z}$ and

$$\hat{\alpha}(\hat{Z}) = \sup_{B \in \mathcal{A}_\rho} \left\{ E\left[\hat{Z}_{0,T} B\right] \right\} = \frac{1}{\hat{Z}_0} \sup_{B \in \mathcal{A}_\rho} \ell(B) = \frac{1}{\hat{Z}_0} \beta,$$

thus

$$\hat{\alpha}(\hat{Z}) < \frac{1}{\hat{Z}_0} \ell(B - m) = \frac{1}{\hat{Z}_0} E_{Q_0}[B\xi] - m = E[B\hat{Z}_{0,T}] - m,$$

which is a contradiction to the choice of m. Thus $B - m \in \mathcal{A}_\rho$ and thus $m \geq \rho(B)$. □

Definition. We call a dynamic convex valuation C **continuous from above at time** t, if for any uniformly bounded sequence $(B_n)_{n\in\mathbb{N}} \subset L^\infty(\mathscr{F}_T)$ decreasing to some $B \in L^\infty(\mathscr{F}_T)$, we have

$$\lim_{n\to\infty} C_t(B_n) = C_t(B).$$

By a similar argument as in Theorem 3.13 of Klöppel and Schweizer(2007)[19], we can state the following theorem

Theorem 1.1. *If $C = \{C(B) = (C_t(B))_{t\in[0,T]}, B \in L^\infty(\mathscr{F}_T)\}$ is a dynamic convex valuation related to S, which is continuous from above at time t, there exists a "penalty functional" α such that for all $B \in L^\infty(\mathscr{F}_T)$*

$$C_t(B) = \operatorname*{ess\,sup}_{Z\in\mathbb{Z}_e} \left\{ E\left[Z_{t,T}B \middle| \mathscr{F}_t \right] - \alpha_t(Z_{t,T}) \right\},$$

where $Z_{t,T} = \frac{Z_T}{Z_t}$ and

$$\alpha_t(Z_{t,T}) := \operatorname*{ess\,sup}_{B\in L^\infty(\mathscr{F}_T)} \left\{ E\left[Z_{t,T}B \middle| \mathscr{F}_t \right] - C_t(B) \right\}$$

is the convex conjugate of C_t.

Proof. For every $Z \in \mathbb{Z}_e$ and $B \in L^\infty(\mathscr{F}_T)$,

$$\begin{aligned} E\left[Z_{t,T}B \middle| \mathscr{F}_t \right] - \alpha_t(Z_{t,T}) &= E\left[Z_{t,T}B \middle| \mathscr{F}_t \right] - \operatorname*{ess\,sup}_{B\in L^\infty(\mathscr{F}_T)} \left\{ E\left[Z_{t,T}B \middle| \mathscr{F}_t \right] - C_t(B) \right\} \\ &\le E\left[Z_{t,T}B \middle| \mathscr{F}_t \right] - \left\{ E\left[Z_{t,T}B \middle| \mathscr{F}_t \right] - C_t(B) \right\} \\ &= C_t(B), \end{aligned}$$

thus

$$C_t(B) \ge \operatorname*{ess\,sup}_{Z\in\mathbb{Z}_e} \left\{ E\left[Z_{t,T}B \middle| \mathscr{F}_t \right] - \alpha_t(Z_{t,T}) \right\} .$$

We only need to prove that for all $B \in L^\infty(\mathscr{F}_T)$

$$E_{Q^0}[C_t(B)] \le E_{Q^0} \left[\operatorname*{ess\,sup}_{Z\in\mathbb{Z}_e} \left\{ E\left[Z_{t,T}B \middle| \mathscr{F}_t \right] - \alpha_t(Z_{t,T}) \right\} \right] . \tag{3.1}$$

Let $\rho_t(B) = E_{Q^0}[C_t(B)]$, then $\rho_t(B)$ is continuous from above and thus $\mathcal{A}'_t := \{B; \rho_t(B) \le 0\}$ is $\sigma(L^\infty(Q^0, \mathscr{F}_T), L^1(Q^0, \mathscr{F}_T))$-closed. Therefore, ρ_t can be represented into the following form

$$\rho_t(B) = \sup_{Z\in\mathbb{Z}} \{E[Z_{0,T}B] - \hat{\alpha}_t(Z)\},$$

where $\hat{\alpha}_t$ is given by

$$\hat{\alpha}_t(Z) = \sup_{B\in L^\infty(\mathscr{F}_T)} \{E(Z_{0,T}B) - \rho_t(B)\}.$$

For $Z \in \mathbb{Z}_e$ let

$$\hat{Z}_s = Z^0_s I_{s<t} + Z^0_t Z_{t,s} I_{s>t},$$

and $\mathbb{Z}_t = \{\hat{Z}; Z \in \mathbb{Z}_e\}$. So $\mathbb{Z}_t \subseteq \mathbb{Z}_e \subset \mathbb{Z}$ and

$$\rho_t(B) \ge \sup_{Z\in\mathbb{Z}_t} \{E[Z_{0,T}B] - \hat{\alpha}_t(Z)\}.$$

For any $Z \in \mathbb{Z}$ with $Z \notin \mathbb{Z}_t$, there exists an $\mathscr{F}_t$-measurable set A such that $E(Z_{0,t}I_A) \neq E(Z_t^0 I_A)$. Thus for any constant λ, $\rho_t(\lambda I_A) = E_{Q^0}(C_t(\lambda I_A)) = \lambda E(Z_t^0 I_A)$ and

$$\begin{aligned}\hat{\alpha}_t(Z) &\geq \sup_{\lambda\in\mathbb{R}}\{\lambda E(Z_{0,T}I_A) - \rho_t(\lambda I_A)\}\\ &= \sup_{\lambda\in\mathbb{R}}\left\{\lambda\big(E(Z_{0,t}I_A) - E(Z_t^0 I_A)\big)\right\} = \infty.\end{aligned}$$

Therefore,

$$\rho_t(B) = \sup_{Z\in\mathbb{Z}_t}\{E[Z_{0,T}B] - \hat{\alpha}_t(Z)\}. \tag{3.2}$$

From the $\mathscr{F}_t$-regularity of C_t, $\{E(Z_{t,T}B|\mathscr{F}_t) - C_t(B); B \in L^\infty(\mathscr{F}_T)\}$ is closed by $\vee$, thus there exists a sequence $(B_n)_{n\in\mathbb{N}} \subseteq L^\infty(\mathscr{F}_T)$ such that

$$\begin{aligned}\alpha_t(Z_{t,T}) &= \operatorname*{ess\,sup}_{B\in L^\infty(\mathscr{F}_T)}\{E(Z_{t,T}B_n|\mathscr{F}_t) - C_t(B_n)\}\\ &=\nearrow \lim_{n\to\infty} E(Z_{t,T}B_n|\mathscr{F}_t) - C_t(B_n).\end{aligned}$$

Since $Z_{0,t} = Z_t^0$ for all $Z \in \mathbb{Z}_t$, it follows from the monotone convergence theorem that

$$\begin{aligned}E[Z_t^0\alpha_t(Z_{t,T})] &= E[Z_{0,t}\alpha_t(Z_{t,T})]\\ &=\nearrow \lim_{n\to\infty} E\left\{Z_{0,t}\big(E(Z_{t,T}B_n|\mathscr{F}_t) - C_t(B_n)\big)\right\}\\ &\leq \sup_{B\in L^\infty(\mathscr{F}_T)}\left\{E(Z_{0,T}B) - E_{Q^0}[C_t(B)]\right\}\\ &= \hat{\alpha}_t(Z).\end{aligned}$$

Combining with (3.2), one gets

$$\begin{aligned}E_{Q^0}[C_t(B)] &= \rho_t(B) = \sup_{Z\in\mathbb{Z}_t}\{E[Z_{0,T}B] - \hat{\alpha}_t(Z)\}\\ &\leq \sup_{Z\in\mathbb{Z}_t}\{E[Z_t^0 Z_{t,T}B] - E[Z_t^0\alpha_t(Z_{t,T})]\}\\ &= \sup_{Z\in\mathbb{Z}_t}\left\{E_{Q^0}\left[E(Z_{t,T}B|\mathscr{F}_t) - \alpha_t(Z_{t,T})\right]\right\}\\ &\leq E_{Q^0}\left[\operatorname*{ess\,sup}_{Z\in\mathbb{Z}_e}\left\{E(Z_{t,T}B|\mathscr{F}_t) - \alpha_t(Z_{t,T})\right\}\right],\end{aligned}$$

and hence (3.1). □

Remark 3.2.

1. *Since for $\pi \in Adm$, $C_t(X_T^{x,\pi}) = X_t^{x,\pi}$, so*

$$\begin{aligned}X_t^{x,\pi} &= C_t(X_T^{x,\pi})\\ &= \operatorname*{ess\,sup}_{Z\in\mathbb{Z}_e}\left\{E\left[Z_{t,T}X_T^{x,\pi}|\mathscr{F}_t\right] - \alpha_t(Z_{t,T})\right\}\\ &= X_t^{x,\pi} - \operatorname*{ess\,inf}_{Z\in\mathbb{Z}_e}\{\alpha_t(Z_{t,T})\}.\end{aligned}$$

So

$$\operatorname*{ess\,inf}_{Z\in\mathbb{Z}_e}\{\alpha_t(Z_{t,T})\} = 0\ . \tag{3.3}$$

2. *From the proof of Theorem 1.1, one can see that for every $Z \in \mathbb{Z}_e$, and there exists a sequence $(B_n^t)_{n\in\mathbb{N}} \subseteq L^\infty(\mathscr{F}_T)$,which may depend on t, such that*

$$\alpha_t(Z_{t,T}) = \nearrow \lim_{n\to\infty} E(Z_{t,T}B_n^t|\mathscr{F}_t) - C_t(B_n^t). \tag{3.4}$$

3. *Let $\mathcal{A}_t := \{B; C_t(B) \le 0\}$, then*

$$\alpha_t(Z_{t,T}) = \operatorname*{ess\,sup}_{B\in\mathcal{A}_t} \left\{E\left[Z_{t,T}B\middle|\mathscr{F}_t\right]\right\} . \tag{3.5}$$

By the time-consistency of C, $\mathcal{A}_t \subseteq \mathcal{A}_s$ for all $s \le t$. As $C_t(C_t(B)) = C_t(B)$, we have for all $s \le t$

$$C_s(C_t(B)) = C_s(B),$$

and similar to [19], one can show that the time-consistency is equivalent to the following

$$\mathcal{A}_s = A_s(\mathscr{F}_t) + \mathcal{A}_t, \tag{3.6}$$

where $\mathcal{A}_s(\mathscr{F}_t) = \mathcal{A}_s \cap L^\infty(\mathscr{F}_\infty)$.

For any $Z^i \in \mathbb{Z}_e$, $i = 1,2$ and $A \in \mathscr{F}_t$, let

$$Z_s := Z_s^1 I_{s<t} + \{Z_s^1 I_A + Z_t^1 Z_{t,s}^2 I_{A^c}\} I_{s\ge t},$$

one can see that $Z \in \mathbb{Z}_e$ and $Z_{t,T} = Z_{t,T}^1 I_A + Z_{t,T}^2 I_{A^c}$. We have the following Lemma

Lemma 3.3. *Under the conditions of Theorem 1.1, for any $Z^i \in \mathbb{Z}_e$, $i = 1,2$ and $A \in \mathscr{F}_t$, we have*

$$\alpha_t(Z_{t,T}^1 I_A + Z_{t,T}^2 I_{A^c}) = \alpha_t(Z_{t,T}^1) I_A + \alpha_t(Z_{t,T}^2) I_{A^c} .$$

Proof. By direct calculation, one can see

$$\begin{aligned}
\alpha_t(Z_{t,T}^1 I_A + Z_{t,T}^2 I_{A^c}) &= \operatorname*{ess\,sup}_{B\in L^\infty(\mathscr{F}_T)} \left\{E\left[(Z_{t,T}^1 I_A + Z_{t,T}^2 I_{A^c})B\middle|\mathscr{F}_t\right] - C_t(B)\right\} \\
&= \operatorname*{ess\,sup}_{B\in L^\infty(\mathscr{F}_T)} \left\{\left\{E[Z_{t,T}^1 B|\mathscr{F}_t] - C_t(B)\right\} I_A + \left\{E[Z_{t,T}^2 B|\mathscr{F}_t] - C_t(B)\right\} I_{A^c}\right\} \\
&\le \operatorname*{ess\,sup}_{B\in L^\infty(\mathscr{F}_T)} \left\{E\left[Z_{t,T}^1 B\middle|\mathscr{F}_t\right] - C_t(B)\right\} I_A + \operatorname*{ess\,sup}_{B\in L^\infty(\mathscr{F}_T)} \left\{E\left[Z_{t,T}^2 B\middle|\mathscr{F}_t\right] - C_t(B)\right\} I_{A^c} \\
&= \alpha_t(Z_{t,T}^1) I_A + \alpha_t(Z_{t,T}^2) I_{A^c}.
\end{aligned}$$

On the other hand, for any $B^i \in L^\infty(\mathscr{F}_T)$, $i = 1,2$, let $B = B_1 I_A + B_2 I_{A^c}$, which also belongs to $L^\infty(\mathscr{F}_T)$. It follows from the $\mathscr{F}_t$-regularity of C_t that

$$\begin{aligned}
\alpha_t(Z_{t,T}^1 I_A + Z_{t,T}^2 I_{A^c}) &\ge E\left[(Z_{t,T}^1 I_A + Z_{t,T}^2 I_{A^c})B\middle|\mathscr{F}_t\right] - C_t(B) \\
&= \left\{E[Z_{t,T}^1 B^1|\mathscr{F}_t] - C_t(B^1)\right\} I_A + \left\{E[Z_{t,T}^2 B|\mathscr{F}_t] - C_t(B)\right\} I_{A^c},
\end{aligned}$$

thus $\alpha_t(Z_{t,T}^1 I_A + Z_{t,T}^2 I_{A^c}) \ge \alpha_t(Z_{t,T}^1) I_A + \alpha_t(Z_{t,T}^2) I_{A^c}$ and the proof is complete. □

For any $Z^i \in \mathbb{Z}_e$, $i = 1, 2$, let

$$A = \left\{ E(Z^1_{t,T} B \big| \mathscr{F}_t) - \alpha_t(Z^1_{t,T}) \geq E(Z^2_{t,T} B \big| \mathscr{F}_t) - \alpha_t(Z^2_{t,T}) \right\},$$

one can see that $A \in \mathscr{F}_t$ and

$$Z_s := Z^1_s I_{s<t} + \{ Z^1_s I_A + Z^1_t Z^2_{t,s} I_{A^c} \} I_{s \geq t}$$

also belongs to $\mathbb{Z}_e$ and $Z_{t,T} = Z^1_{t,T} I_A + Z^2_{t,T} I_{A^c}$. It follows from Lemma 3.3 that $\alpha_t(Z_{t,T}) = \alpha_t(Z^1_{t,T}) I_A + \alpha_t(Z^2_{t,T}) I_{A^c}$ and thus

$$\begin{aligned} E(Z_{t,T} B \big| \mathscr{F}_t) - \alpha_t(Z_{t,T}) &= \left\{ E(Z^1_{t,T} B \big| \mathscr{F}_t) - \alpha_t(Z^1_{t,T}) \right\} I_A + \left\{ E(Z^2_{t,T} B \big| \mathscr{F}_t) - \alpha_t(Z^2_{t,T}) \right\} I_{A^c} \\ &= \left\{ E(Z^1_{t,T} B \big| \mathscr{F}_t) - \alpha_t(Z^1_{t,T}) \right\} \bigvee \left\{ E(Z^2_{t,T} B \big| \mathscr{F}_t) - \alpha_t(Z^2_{t,T}) \right\}, \end{aligned}$$

from which one can see that the set $\left\{ E(Z_{t,T} B \big| \mathscr{F}_t) - \alpha_t(Z_{t,T}); Z \in \mathbb{Z}_e \right\}$ is closed under $\vee$, thus we have the following corollary

Corollary 3.4. *Under the conditions of Theorem 1.1, for every $B \in L^\infty(\mathscr{F}_T)$, there exists a sequence $(Z^n)_{n \in \mathbb{N}} \subseteq \mathbb{Z}_e$ such that*

$$C_t(B) = \nearrow \lim_{n \to \infty} E(Z^n_{t,T} B \big| \mathscr{F}_t) - \alpha_t(Z^n_{t,T}).$$

Lemma 3.5. *Under the conditions of Theorem 1.1, for all $Z \in \mathbb{Z}_e$ and $s < t$, we have*

$$Z_s \alpha_s(Z_{s,T}) = E\left[Z_t \alpha_t(Z_{t,T}) \big| \mathscr{F}_s \right] + \operatorname*{ess\,sup}_{B \in \mathcal{A}_s(\mathscr{F}_t)} E\left[Z_t B \big| \mathscr{F}_s \right], \tag{3.7}$$

where $\mathcal{A}_s(\mathscr{F}_t) = \mathcal{A}_s \cap L^\infty(\mathscr{F}_t)$.

Proof. It follows from (3.6) that any $B \in \mathcal{A}_s$ can be decomposed into the following

$$B = B_1 + B_2,$$

where $B_1 \in \mathcal{A}_s(\mathscr{F}_t)$ and $B_2 \in \mathcal{A}_t$. Therefore,

$$\begin{aligned} E\left[Z_T B \big| \mathscr{F}_s \right] &= E\left[Z_t B_1 \big| \mathscr{F}_s \right] + E\left[E[Z_T B_2 \big| \mathscr{F}_t] \big| \mathscr{F}_s \right] \\ &\leq \operatorname*{ess\,sup}_{B \in \mathcal{A}_s(\mathscr{F}_t)} E\left[Z_t B \big| \mathscr{F}_s \right] + E\left[\operatorname*{ess\,sup}_{B \in \mathcal{A}_t} E[Z_T B \big| \mathscr{F}_t] \bigg| \mathscr{F}_s \right] \\ &= \operatorname*{ess\,sup}_{B \in \mathcal{A}_s(\mathscr{F}_t)} E\left[Z_t B \big| \mathscr{F}_s \right] + E\left[Z_t \alpha_t(Z_{t,T}) \big| \mathscr{F}_s \right], \end{aligned}$$

thus $Z_s \alpha_s(Z_{s,T}) \leq E\left[Z_t \alpha_t(Z_{t,T}) \big| \mathscr{F}_s \right] + \operatorname*{ess\,sup}_{B \in \mathcal{A}_s(\mathscr{F}_t)} E\left[Z_t B \big| \mathscr{F}_s \right]$. On the other hand, there exists a sequence $B^t_n \in L^\infty$ such that

$$\alpha_t(Z_{t,T}) = \nearrow \lim_{n \to \infty} E[Z_{t,T} B^t_n | \mathscr{F}_t].$$

We need to show that

$$E[Z_t\alpha_t(Z_{t,T})\big|\mathscr{F}_s] = \nearrow \lim_{n\to\infty} E\left[Z_T B_n^t\big|\mathscr{F}_s\right] . \tag{3.8}$$

It is easy to see that $\left\{E\left[Z_T B_n^t\big|\mathscr{F}_s\right]; n\in\mathbb{N}\right\}$ is an increasing sequence and for each n

$$E\left[Z_T B_n^t\big|\mathscr{F}_s\right] \le E[Z_t\alpha_t(Z_{t,T})\big|\mathscr{F}_s].$$

Since $B_1^t \in L^\infty(\mathscr{F}_T)$, there exists a constant c such that $B_1^t \ge c$, thus for each $A\in\mathscr{F}_s$

$$E\left[E[Z_T B_1^t|\mathscr{F}_t]I_A\right] = E\left[E[Z_T B_1^t|\mathscr{F}_s]I_A\right] \ge cE[Z_s I_A] > -\infty$$

From the monotone convergence theorem we get

$$\begin{aligned} E[Z_t\alpha_t(Z_{t,T})I_A] &= \lim_{n\to\infty} E\left[E[Z_T B_n^t|\mathscr{F}_t]I_A\right] \\ &= \lim_{n\to\infty} E\left[E[Z_T B_n^t|\mathscr{F}_s]I_A\right] \\ &= E\left[\nearrow \lim_{n\to\infty} E[Z_T B_n^t|\mathscr{F}_s]I_A\right] \end{aligned}$$

holds for every $A\in\mathscr{F}_s$, thus $E[Z_t\alpha_t(Z_{t,T})|\mathscr{F}_s] = \nearrow \lim_{n\to\infty} E[Z_T B_n^t|\mathscr{F}_s]$, which is (3.8). For each $B_1\in\mathcal{A}_s(\mathscr{F}_t)$, let $B^n := B_1 + B_n^t \in \mathcal{A}_s$, thus

$$\begin{aligned} E\left[Z_t B_1\big|\mathscr{F}_s\right] + E\left[Z_t\alpha_t(Z_{t,T})\big|\mathscr{F}_s\right] &= E\left[Z_T B_1\big|\mathscr{F}_s\right] + \nearrow \lim_{n\to\infty} E\left[Z_T B_n^t\big|\mathscr{F}_s\right] \\ &= \nearrow \lim_{n\to\infty} E\left[Z_T B^n\big|\mathscr{F}_s\right] \\ &\le \operatorname*{ess\,sup}_{B\in\mathcal{A}_s} E\left[Z_T B\big|\mathscr{F}_s\right] \\ &= Z_s\alpha_s(Z_{s,T}). \end{aligned}$$

The arbitrariness of B_1 gives

$$E\left[Z_t\alpha_t(Z_{t,T})\big|\mathscr{F}_s\right] + \operatorname*{ess\,sup}_{B\in\mathcal{A}_s(\mathscr{F}_t)} E\left[Z_t B\big|\mathscr{F}_s\right] Z_s \le \alpha_s(Z_{s,T})$$

and the proof is complete. □

Corollary 3.6. *Under the conditions of Theorem 1.1, for every $Z\in\mathbb{Z}_e$, $\{Z_t\alpha_t(Z_{t,T})\}_{t\in[0,T]}$ is a nonnegative supermartingale with terminal value $Z_T\alpha_T(Z_{T,T}) = 0$.*

Remark 3.7. *Under the conditions of Theorem 1.1, for all $Z\in\mathbb{Z}_e$ and $s,t\in[0,T]$ with $s<t$, let $\alpha_s(Z_{s,t}) := \operatorname{ess\,sup}_{B\in\mathcal{A}_s(\mathscr{F}_t)} E\left[Z_{s,t}B\big|\mathscr{F}_s\right]$, then (3.7) can be rewritten as*

$$Z_s\alpha_s(Z_{s,T}) = E\left[Z_t\alpha_t(Z_{t,T})\big|\mathscr{F}_s\right] + Z_s\alpha_s(Z_{s,t}).$$

We have the following theorem

Theorem 3.8. *Given a family of convex nonnegative functionals*

$$\alpha = \left\{\alpha(Z) = \left(\alpha_s(Z_{s,t})\right)_{0\le s<t\le T};\ Z\in\mathbb{Z}_e\right\}$$

such that $\alpha_{s,t} : \mathbb{Z}_e \longrightarrow L^0(\mathscr{F}_s,[0,\infty))$ with $E[Z_{s,t}\alpha_t(Z_{t,T})] < \infty$ for any $Z\in\mathbb{Z}_e$ and $0\le s< t\le T$ satisfies (*i*) $\alpha_s(Z_{s,T}) = \alpha_s(Z_{s,t}) + E[Z_{s,t}\alpha_t(Z_{t,T})|\mathscr{F}_s]$, *for each* $Z \in \mathbb{Z}_e$ *and* $0 \le s < t \le T$*;*

(*ii*) $\operatorname*{ess\,inf}_{Z\in\mathbb{Z}_e} \{\alpha_t(Z_{t,T})\} = 0$, *for each* $t \in [0, T]$*;*

(*iii*) *for any* $Z^i \in \mathbb{Z}_e$, $i = 1, 2$ *and* $A \in \mathscr{F}_t$, *let* $Z_s = Z^1_s I_{s<t} + Z^1_t\{Z^1_{t,s} I_A + Z^2_{t,s} I_{A^c}\} I_{s\ge t}$, *then* $\alpha_t(Z_{t,T}) = \alpha_t(Z^1_{t,T}) I_A + \alpha_t(Z^2_{t,T}) I_{A^c}$*;*

(*iv*) $\lim_{h\searrow 0} \alpha_t(Z_{t,t+h}) = 0$, *for* $Z \in \mathbb{Z}_e$ *and* $t \in [0, T)$,

then $C = \{C(B) = (C_t(B))_{t\in[0,T]}, B \in L^\infty(\mathscr{F}_T)\}$ *defined by*

$$C_t(B) = \operatorname*{ess\,sup}_{Z\in\mathbb{Z}_e} \left\{E\left[Z_{t,T}B\middle|\mathscr{F}_t\right] - \alpha_t(Z_{t,T})\right\} \tag{3.9}$$

is an S*-related dynamic convex valuation.*

Remark 3.9. 1) *It follows from (iii) that for any* $Z^i \in \mathbb{Z}_e$, $i = 1, 2$ *and* $\hat{Z}_u = Z^1_u I_{u<t} + Z^1_t Z^2_{t,u} I_{u\ge t}$, *then*

$$\alpha_t(\hat{Z}_{t,T}) = \alpha_t(Z^2_{t,T}).$$

2) *From (i) and (ii), for any* $s < t$

$$0 = \operatorname*{ess\,inf}_{Z\in\mathbb{Z}_e} \{\alpha_s(Z_{s,T})\} \ge \operatorname*{ess\,inf}_{Z\in\mathbb{Z}_e} \{\alpha_s(Z_{s,t})\} + \operatorname*{ess\,inf}_{Z\in\mathbb{Z}_e} E[Z_{s,t}\alpha_t(Z_{t,T})|\mathscr{F}_s],$$

which implies that

$$\operatorname*{ess\,inf}_{Z\in\mathbb{Z}_e} \{\alpha_s(Z_{s,t})\} = 0 \quad \textit{and} \quad \operatorname*{ess\,inf}_{Z\in\mathbb{Z}_e} E[Z_{s,t}\alpha_t(Z_{t,T})|\mathscr{F}_s] = 0.$$

3) *Since* $\left\{E\left[Z_{t,T}B\middle|\mathscr{F}_t\right] - \alpha_t(Z_{t,T}); Z \in \mathbb{Z}_e\right\}$ *is closed under* $\vee$, *there exists a sequence* $(Z^n)_{n\in\mathbb{N}} \subset \mathbb{Z}_e$ *such that*

$$C_t(B) = \nearrow \lim_{n\to\infty} E\left[Z^n_{t,T}B\middle|\mathscr{F}_t\right] - \alpha_t(Z^n_{t,T}).$$

4) *Under the conditions of Theorem 3.8, for any* $s < t$ *and* $B \in L^\infty(\mathscr{F}_t)$, *we have*

$$C_s(B) = \operatorname*{ess\,sup}_{Z\in\mathbb{Z}_e} \left\{E\left[Z_{s,t}B\middle|\mathscr{F}_s\right] - \alpha_s(Z_{s,t})\right\}. \tag{3.10}$$

In fact,

$$\begin{aligned} C_s(B) &= \operatorname{ess\,sup}_{Z\in\mathbb{Z}_e} \left\{E\left[Z_{s,T}B\middle|\mathscr{F}_s\right] - \alpha_s(Z_{s,T})\right\} \\ &\le \operatorname{ess\,sup}_{Z\in\mathbb{Z}_e} \left\{E\left[Z_{s,t}B\middle|\mathscr{F}_s\right] - \alpha_s(Z_{s,t})\right\}, \end{aligned}$$

since $\alpha_s(Z_{s,T}) \ge \alpha_s(Z_{s,t}) \ge 0$. *On the other hand, for any* $Z^i \in \mathbb{Z}_e$, $i = 1, 2$, *let* $\hat{Z}_u = Z^1_u I_{u<t} + Z^1_t Z^2_{t,u} I_{u\ge t}$, *it follows from (i) that*

$$\begin{aligned} C_s(B) &\ge E\left(\hat{Z}_{s,T}B\middle|\mathscr{F}_s\right) - \alpha_s(\hat{Z}_{s,T}) \\ &= E\big[\hat{Z}_{s,t}B\big|\mathscr{F}_s\big] - \alpha_s(\hat{Z}_{s,t}) - E\big[\hat{Z}_{s,t}\alpha_t(\hat{Z}_{t,T})\big|\mathscr{F}_s\big] \\ &= E\big(Z^1_{s,t}B\big|\mathscr{F}_s\big) - \alpha_s(Z^1_{s,t}) - E\big(Z^1_{s,t}\alpha_t(Z^2_{t,T})\big|\mathscr{F}_s\big), \end{aligned}$$

holds for all $Z^2 \in \mathbb{Z}_e$*, thus*

$$\begin{aligned} C_s(B) &\geq E\left(Z^1_{s,t}B \middle| \mathscr{F}_s\right) - \alpha_s(Z^1_{s,t}) - E\big[Z^1_{s,t} \operatorname*{ess\,inf}_{Z\in\mathbb{Z}_e} \alpha_t(Z_{t,T}) \big| \mathscr{F}_s\big] \\ &= E\left(Z^1_{s,t}B \middle| \mathscr{F}_s\right) - \alpha_s(Z^1_{s,t}), \end{aligned}$$

therefore $C_s(B) \geq \operatorname*{ess\,sup}_{Z\in\mathbb{Z}_e} \left\{E\left[Z_{s,t}B \middle| \mathscr{F}_s\right] - \alpha_s(Z_{s,t})\right\}$*, and (3.10) follows.*

Lemma 3.10. *Under the conditions of Theorem 3.8, for any* $B \in L^\infty(\mathscr{F}_T)$*,* $C(B)$ *defined by* (3.9) *is a bounded RCLL semimartingale such that* $\left\{\hat{Z}_t\big(C_t(B) + \alpha_t(\hat{Z}_{t,T})\big);\ t \in [0,T]\right\}$ *is a supermartingale for any* $\hat{Z} \in \mathbb{Z}_e$ *.*

Proof. For any $\hat{Z} \in \mathbb{Z}_e$, let

$$\mathbb{Z}_t(\hat{Z}) := \left\{Z \in \mathbb{Z}_e;\ ZI_{[\![0,t]\!]} = \hat{Z}I_{[\![0,t]\!]}\right\},$$

so that $\mathbb{Z}_t(\hat{Z}) \subset \mathbb{Z}_s(\hat{Z})$ for every $s < t$, and for every $Z \in \mathbb{Z}_t(\hat{Z})$, Z can be rewritten as

$$Z_u = \hat{Z}_u I_{u<t} + \hat{Z}_t Z_{t,T} I_{u\geq T}$$

and thus $\alpha_s(Z_{s,T}) = \alpha_s(\hat{Z}_{s,t}) + E\left[\hat{Z}_{s,t}\alpha_t(Z_{t,T}) \middle| \mathscr{F}_s\right]$. Therefore,

$$\begin{aligned} \hat{Z}_s\left(C_s(B) + \alpha_s(\hat{Z}_{s,T})\right) &= \hat{Z}_s \operatorname*{ess\,sup}_{Z\in\mathbb{Z}_e} \left\{E\left[Z_{s,T}B \middle| \mathscr{F}_s\right] - \alpha_s(Z_{s,T})\right\} + \hat{Z}_s\alpha_s(\hat{Z}_{s,T}) \\ &= \operatorname*{ess\,sup}_{Z\in\mathbb{Z}_s(\hat{Z})} \left\{E\left[Z_T B \middle| \mathscr{F}_s\right] - \hat{Z}_s\alpha_s(Z_{s,T}) + \hat{Z}_s\alpha_s(\hat{Z}_{s,T})\right\} \\ &\geq \operatorname*{ess\,sup}_{Z\in\mathbb{Z}_t(\hat{Z})} \left\{E\left[Z_T B \middle| \mathscr{F}_s\right] - \hat{Z}_s\alpha_s(Z_{s,T}) + \hat{Z}_s\alpha_s(\hat{Z}_{s,T})\right\} \\ &= \operatorname*{ess\,sup}_{Z\in\mathbb{Z}_t(\hat{Z})} \Big\{E\left[Z_T B \middle| \mathscr{F}_s\right] - \hat{Z}_s\alpha_s(\hat{Z}_{s,t}) - \hat{Z}_s E\big[\hat{Z}_{s,t}\alpha_t(Z_{t,T}) \big| \mathscr{F}_s\big] \\ &\qquad + \hat{Z}_s\alpha_s(\hat{Z}_{s,t}) + \hat{Z}_s E\big[\hat{Z}_{s,t}\alpha_t(\hat{Z}_{t,T}) \big| \mathscr{F}_s\big]\Big\} \\ &= \operatorname*{ess\,sup}_{Z\in\mathbb{Z}_t(\hat{Z})} \left\{E\left[Z_T B \middle| \mathscr{F}_s\right] - E\big[\hat{Z}_t\alpha_t(Z_{t,T}) \big| \mathscr{F}_s\big] + E\big[\hat{Z}_t\alpha_t(\hat{Z}_{t,T}) \big| \mathscr{F}_s\big]\right\}. \end{aligned}$$

From Remark 3.9 3), one can choose a sequence $(Z^n)_{n\in\mathbb{N}} \subset \mathbb{Z}_t(\hat{Z})$ such that

$$C_t(B) = \nearrow \lim_{n\to\infty} E[Z^n_{t,T}B|\mathscr{F}_t] - \alpha_t(Z^n_{t,T}) .$$

It follows from the monotone convergence theorem that

$$\begin{aligned} \hat{Z}_s\left(C_s(B) + \alpha_s(\hat{Z}_{s,T})\right) &\geq \operatorname*{ess\,sup}_{Z\in\mathbb{Z}_t(\hat{Z})} \left\{E\left[Z_T B \middle| \mathscr{F}_s\right] - E\big[\hat{Z}_t\alpha_t(Z_{t,T}) \big| \mathscr{F}_s\big] + E\big[\hat{Z}_t\alpha_t(\hat{Z}_{t,T}) \big| \mathscr{F}_s\big]\right\} \\ &\geq \lim_{n\to\infty} E\left\{\hat{Z}_t\left\{E\big[Z^n_{t,T}B \big| \mathscr{F}_t\big] - \alpha_t(Z^n_{t,T})\right\} \middle| \mathscr{F}_s\right\} + E\big[\hat{Z}_t\alpha_t(\hat{Z}_{t,T}) \big| \mathscr{F}_s\big] \\ &= E\left\{\hat{Z}_t \nearrow \lim_{n\to\infty} \left\{E\big[Z^n_{t,T}B \big| \mathscr{F}_t\big] - \alpha_t(Z^n_{t,T})\right\} \middle| \mathscr{F}_s\right\} + E\big[\hat{Z}_t\alpha_t(\hat{Z}_{t,T}) \big| \mathscr{F}_s\big] \\ &= E\left\{\hat{Z}_t\big\{C_t(B) + \alpha_t(\hat{Z}_{t,T})\big\} \middle| \mathscr{F}_s\right] , \end{aligned}$$

thus $\left\{\hat{Z}_t\big(C_t(B)+\alpha_t(\hat{Z}_{t,T})\big);\ t\in[0,T]\right\}$ is a supermartingale. Similar to El Karoui and Quenez (1995)[11] or Laurent and Pham (1999)[22], one can show that $\left\{\hat{Z}_t\big(C_t(B)+\alpha_t(\hat{Z}_{t,T})\big);\ t\in[0,T]\right\}$ has an RCLL version. As $\hat{Z}\alpha(\hat{Z}):=\{\hat{Z}_t\alpha_t(\hat{Z}_{t,T}); t\in[0,T]\}$ is also an RCLL supermartingale, thus $C(B)$ has an RCLL version. □

Corollary 3.11. *Under the conditions of Theorem 3.8, if there exists $\hat{Z}\in\mathbb{Z}_e$ such that $\hat{Z}_t(C_t(B)+\alpha_t(\hat{Z}_{t,T}))$ is a uniformly integrable martingale, then for every $t\in[0,T]$*

$$C_t(B)=E\left[\hat{Z}_{t,T}B\middle|\mathscr{F}_t\right]-\alpha_t(\hat{Z}_{t,T}),\ a.s.$$

Proof of Theorem 3.8. One can easily check that C defined by (3.9) satisfies (A), (B) and (C) in the definition of the S-related DCV. For any $A\in\mathscr{F}_t$ and $B_i\in L^\infty(\mathscr{F}_T)$, $i=1,2$,

$$\begin{aligned}
&E\left[Z_{t,T}(B_1I_A+B_2I_{A^c})\middle|\mathscr{F}_t\right]-\alpha_t(Z_{t,T})\\
&\quad=\left\{E\left[Z_{t,T}B_1\middle|\mathscr{F}_t\right]-\alpha_t(Z_{t,T})\right\}I_A+\left\{E\left[Z_{t,T}B_2\middle|\mathscr{F}_t\right]-\alpha_t(Z_{t,T})\right\}I_{A^c}\\
&\quad\le \operatorname*{ess\,sup}_{Z\in\mathbb{Z}_e}\left\{E\left[Z_{t,T}B_1\middle|\mathscr{F}_t\right]-\alpha_t(Z_{t,T})\right\}I_A+\operatorname*{ess\,sup}_{Z\in\mathbb{Z}_e}\left\{E\left[Z_{t,T}B_2\middle|\mathscr{F}_t\right]-\alpha_t(Z_{t,T})\right\}I_{A^c}\\
&\quad=C_t(B_1)I_A+C_t(B_2)I_{A^c},
\end{aligned}$$

thus $C_t(B_1I_A+B_2I_{A^c})\le C_t(B_1)I_A+C_t(B_2)I_{A^c}$. On the other hand, for any $Z^i\in\mathbb{Z}_e$, $i=1,2$, let $\hat{B}=B_1I_A+B_2I_{A^c}$ and $\hat{Z}_s=Z^1_sI_{s<t}+Z^1_t\{Z^1_{t,s}I_A+Z^2_{t,s}I_{A^c}\}I_{s\ge t}$, from (iii), one gets $\alpha_t(\hat{Z}_{t,T})=\alpha_t(Z^1_{t,T})I_A+\alpha_t(Z^2_{t,T})I_{A^c}$ and $\hat{Z}_{t,T}=Z^1_{t,T}I_A+Z^2_{t,T}I_{A^c}$. Therefore,

$$\begin{aligned}
C_t(\hat{B})&=\operatorname*{ess\,sup}_{Z\in\mathbb{Z}_e}\left\{E\left[Z_{t,T}B\middle|\mathscr{F}_t\right]-\alpha_t(Z_{t,T})\right\}\\
&\ge E\left[\hat{Z}_{t,T}\hat{B}\middle|\mathscr{F}_t\right]-\alpha_t(\hat{Z}_{t,T})\\
&=\left\{E\left[Z^1_{t,T}B_1\middle|\mathscr{F}_t\right]-\alpha_t(Z^1_{t,T})\right\}I_A+\left\{E\left[Z^2_{t,T}B_2\middle|\mathscr{F}_t\right]-\alpha_t(Z^2_{t,T})\right\}I_{A^c},
\end{aligned}$$

thus $C_t(B_1I_A+B_2I_{A^c})\ge C_t(B_1)I_A+C_t(B_2)I_{A^c}$, which implies that $C_t(B)$ satisfies (D), i.e., the $\mathscr{F}_t$-regularity. For any $\pi\in Adm$ and $t\in[0,T]$, it follows from (ii) that

$$C_t(X^{0,\pi}_T)=\operatorname*{ess\,sup}_{Z\in\mathbb{Z}_e}\left\{E\left[Z_{t,T}X^{0,\pi}_T\middle|\mathscr{F}_t\right]-\alpha_t(Z_{t,T})\right\}=X^{0,\pi}_t-\operatorname*{ess\,inf}_{Z\in\mathbb{Z}_e}\{\alpha_t(Z_{t,T})\}=X^{0,\pi}_t,$$

and the S-related property of C follows. We now show the time consistency of C, i.e., for any $s<t$ and $B\in L^\infty(\mathscr{F}_T)$, it follows from (i) and (3.10) that

$$\begin{aligned}
C_s(C_t(B))&=\operatorname*{ess\,sup}_{Z\in\mathbb{Z}_e}\left\{E\left[Z_{s,t}C_t(B)\middle|\mathscr{F}_s\right]-\alpha_s(Z_{s,t})\right\}\\
&\ge\operatorname*{ess\,sup}_{Z\in\mathbb{Z}_e}\left\{E\left[Z_{s,t}E(Z_{t,T}B|\mathscr{F}_t)-Z_{s,t}\alpha_t(Z_{t,T})\middle|\mathscr{F}_s\right]-\alpha_s(Z_{s,t})\right\}\\
&=\operatorname*{ess\,sup}_{Z\in\mathbb{Z}_e}\left\{E\left[Z_{s,t}Z_{t,T}B\middle|\mathscr{F}_s\right]-E\left[Z_{s,t}\alpha_t(Z_{t,T})\middle|\mathscr{F}_s\right]-\alpha_s(Z_{s,t})\right\}\\
&=\operatorname*{ess\,sup}_{Z\in\mathbb{Z}_e}\left\{E\left[Z_{s,T}B\middle|\mathscr{F}_s\right]-\alpha_s(Z_{s,T})\right\}\\
&=C_s(B).
\end{aligned}\tag{3.11}$$

On the other hand, there exists a sequence $(Z^n)_{n\in\mathbb{N}} \subset \mathbb{Z}_e$ such that

$$C_t(B) = \nearrow \lim_{n\to\infty} E\left[Z^n_{t,T} B \middle| \mathscr{F}_t\right] - \alpha_t(Z^n_{t,T})\ ,$$

thus for any $Z \in \mathbb{Z}_e$, let $\hat{Z}^n_u = Z_u I_{u<t} + Z_t Z^n_{t,u} I_{u\geq t}$, then

$$\begin{aligned}
&E\left[Z_{s,t}\left\{E(Z^n_{t,T}B|\mathscr{F}_t) - \alpha_t(Z^n_{t,T})\right\}|\mathscr{F}_s\right] - \alpha_s(Z_{s,t})\\
&\quad = E[Z_{s,t}Z^n_{t,T}B\big|\mathscr{F}_t] - E\left[Z_{s,t}\alpha_t(Z^n_{t,T})\big|\mathscr{F}_s\right] - \alpha_s(Z_{s,t})\\
&\quad = E[\hat{Z}^n_{s,T}B\big|\mathscr{F}_t] - E\left[\hat{Z}^n_{s,t}\alpha_t(\hat{Z}^n_{t,T})\big|\mathscr{F}_s\right] - \alpha_s(\hat{Z}^n_{s,t})\\
&\quad = E[\hat{Z}^n_{s,T}B\big|\mathscr{F}_t] - \alpha_s(\hat{Z}^n_{s,T})\\
&\quad \leq \operatorname*{ess\,sup}_{Z\in\mathbb{Z}_e}\{E[Z_{s,T}B\big|\mathscr{F}_t] - \alpha_s(Z_{s,T})\}\\
&\quad = C_s(B)\ .
\end{aligned}$$

From an analogous argument as in the proof of Lemma 3.5, one can derive

$$E\left[Z_{s,t}C_t(B)\big|\mathscr{F}_s\right] - \alpha_s(Z_{s,t}) = \nearrow \lim_{n\to\infty} E\left[Z_{s,t}\left\{E(Z^n_{t,T}B|\mathscr{F}_t) - \alpha_t(Z^n_{t,T})\right\}|\mathscr{F}_s\right] - \alpha_s(Z_{s,t}) \leq C_s(B),$$

thus

$$C_s(C_t(B)) = \operatorname*{ess\,sup}_{Z\in\mathbb{Z}_e}\left\{E\left[Z_{s,t}C_t(B)\big|\mathscr{F}_s\right] - \alpha_s(Z_{s,t})\right\} \leq C_s(B). \tag{3.12}$$

From (3.11) and (3.12), we have

$$C_s(C_t(B)) = C_s(B)$$

and the time consistency follows. □

4 Two S-related DCVs generated by $\tilde{\alpha}$ and $\hat{\alpha}$

In this section we will consider two special penalty functionals $\tilde{\alpha}$ and $\hat{\alpha}$ satisfying the conditions of Theorem 3.8, and the corresponding S-related dynamic convex valuations generated by $\tilde{\alpha}$ and $\hat{\alpha}$ denoted by $\tilde{C}$ and $\hat{C}$ respectively. We then describe the dynamics of $\tilde{C}$ and $\hat{C}$ by two backward semimartingale equations.

For any $Z \in \mathbb{Z}_e$, it follows from Corollary 2.5 that $\tilde{Z}_t := \dfrac{Z_t}{Z^0_t}$ can be represented in the following form

$$\tilde{Z}_t = Z_0\mathcal{E}\Big\{ -\int_0^\cdot \tilde{l}_u d\widetilde{M}_u + \int_0^\cdot \int_{\mathbb{R}} l(u,y)\tilde{N}(du,dy) + L\Big\}_t,$$

where $l(u, y)$ is a $\widetilde{\mathscr{P}}$-measurable function with $l(u, y) > 1$ and L a continuous local Q^0-martingale strongly Q^0-orthogonal to $\widetilde{M}$ with $L_0 = 0$. For any $s < t$

$$\tilde{Z}_{s,t} = 1 - \int_{]s,t]} \tilde{Z}_{s,u-}\tilde{l}_u d\widetilde{M}_u + \int_{]s,t]}\int_{\mathbb{R}} \tilde{Z}_{s,u-}l(u,y)\tilde{N}(du,dy) + \int_{]s,t]} \tilde{Z}_{s,u-}dL_u,$$

where $\tilde{Z}_{s,t} = \dfrac{\tilde{Z}_t}{\tilde{Z}_s}$ and $\tilde{Z}_{s,u-} = \dfrac{\tilde{Z}_{u-}}{\tilde{Z}_s}$ for $s < u$. Given two predictable processes ξ and η with $0 < c \leq \xi_u \leq C < \infty$ and $0 < c \leq \eta \leq C < \infty$ and a $\widetilde{\mathscr{P}}$-measurable function $\xi(u, y)$ with $0 < c \leq \xi(u, y) \leq C < \infty$, where c and C are two positive constants, we define for each $\tilde{Z} \in \tilde{\mathbb{Z}}_e$

$$\tilde{\alpha}_s(\tilde{Z}_{s,t}) := E_{Q^0}\left(\int_{]s,t]} \tilde{Z}_{s,u-}\xi_u(\tilde{l}_u)^2 dA_u + \int_{]s,t]} \tilde{Z}_{s,u-}\eta_u d\langle L\rangle_u \big| \mathscr{F}_s\right) \tag{4.1}$$

and

$$\hat{\alpha}_s(\tilde{Z}_{s,t}) := E_{Q^0}\left(\int_{]s,t]}\int_{\mathbb{R}} \tilde{Z}_{s,u-}\xi(u,y)l(u,y)^2 K(u;dy)dA_u + \int_{]s,t]} \tilde{Z}_{s,u-}\eta_u d\langle L\rangle_u \big| \mathscr{F}_s\right). \tag{4.2}$$

4.1 The S-related DCV generated by $\tilde{\alpha}$

To consider the S-related dynamic convex valuation generated by $\tilde{\alpha}$, we introduce

$$\mathscr{X} := \left\{\tilde{Z} \in \tilde{\mathbb{Z}}_e;\ \tilde{\alpha}_0(\tilde{Z}_{0,T}) < \infty\right\}$$

Lemma 4.1. *For any $\tilde{Z}^i \in \mathscr{X}$, $i = 1, 2$ and an $\mathscr{F}_t$-measurable set D, let $\tilde{Z}_u = \tilde{Z}^1_u I_{u<t} + \tilde{Z}^1_t\{\tilde{Z}^1_{t,u} I_D + \tilde{Z}^2_{t,u} I_{D^c}\} I_{u\geq t}$, then $\tilde{Z} \in \mathscr{X}$.*

Proof. One can easily see that $\tilde{Z} \in \tilde{\mathbb{Z}}_e$. From

$$E_{Q^0}\left[\tilde{\alpha}_0(\tilde{Z}_{0,T})\right] \leq E_{Q^0}\left[\tilde{\alpha}_0(\tilde{Z}^1_{0,T})\right] + E_{Q^0}\left[\tilde{\alpha}_0(\tilde{Z}^2_{0,T})\right] < \infty,$$

one can see that $\tilde{Z} \in \mathscr{X}$. □

It is easy to see that for each $\tilde{Z} \in \mathscr{X}$ and $s < t$

$$\operatorname*{ess\,inf}_{\tilde{Z}\in\mathscr{X}} \left\{\tilde{\alpha}_s(\tilde{Z}_{s,t})\right\} = 0, \tag{4.3}$$

and

$$\tilde{\alpha}_s(\tilde{Z}_{s,T}) = \tilde{\alpha}_s(\tilde{Z}_{s,t}) + E_{Q^0}[\tilde{Z}_{s,t}\tilde{\alpha}_t(\tilde{Z}_{t,T})\big|\mathscr{F}_s],\ a.s. \tag{4.4}$$

One can easily check that

(iii') for any $\tilde{Z}^i \in \mathscr{X}$, $i = 1, 2$ and a $\mathscr{F}_t$-measurable set D, let $\tilde{Z}_u = \tilde{Z}^1_u I_{u<t} + \tilde{Z}^1_t\{\tilde{Z}^1_{t,u} I_D + \tilde{Z}^2_{t,u} I_{D^c}\} I_{u\geq t}$, then $\tilde{\alpha}_t(\tilde{Z}_{t,T}) = \tilde{\alpha}_t(\tilde{Z}^1_{t,T}) I_D + \tilde{\alpha}_t(\tilde{Z}^2_{t,T}) I_{D^c}$;

(iv') $\lim_{h\searrow 0} \tilde{\alpha}_t(\tilde{Z}_{t,t+h}) = 0$, for any $\tilde{Z} \in \mathscr{X}$ and $t \in [0, T)$.

We have the following lemma

Lemma 4.2. *For any $\tilde{Z}^i \in \mathscr{X}$, $i = 1, 2$ and $\lambda \in [0, 1]$, we have*

$$\tilde{\alpha}_s\left(\lambda\tilde{Z}^1_{s,t} + (1-\lambda)\tilde{Z}^2_{s,t}\right) \leq \lambda\tilde{\alpha}_s(\tilde{Z}^1_{s,t}) + (1-\lambda)\tilde{\alpha}_s(\tilde{Z}^2_{s,t}),\ a.s.$$

Proof. For any $\tilde Z^i\in\mathscr{X}$, $i=1,2$, there exist a $\widetilde{\mathscr{P}}$-measurable function $l^i(u,y)$ and a continuous local martingale L^i with $L^i_0=0$ and $\langle L^i,\tilde M\rangle=0$ such that

$$\tilde Z^i_t=Z^i_0\mathcal{E}\big\{-\int_0^\cdot \widetilde{(l^i)}_u d\widetilde{M}_u+\int_0^\cdot\int_{\mathbb{R}} l^i(u,y)\tilde N(du,dy)+L^i\big\}_t\,.$$

For any $\lambda\in[0,1]$ and $s<t$, one easily derives that $\tilde Z_{s,t}:=\lambda\tilde Z^1_{s,t}+(1-\lambda)\tilde Z^2_{s,t}$ satisfies the following equality

$$\begin{aligned}\tilde Z_{s,t}=1&-\int_s^t\tilde Z_{s,u-}\big\{\frac{\lambda\tilde Z^1_{s,u-}}{\lambda\tilde Z^1_{s,u-}+(1-\lambda)\tilde Z^2_{s,u-}}\widetilde{(l^1)}_u+\frac{(1-\lambda)\tilde Z^2_{s,u-}}{\lambda\tilde Z^1_{s,u-}+(1-\lambda)\tilde Z^2_{s,u-}}\widetilde{(l^2)}_u\big\}d\widetilde{M}_u\\&+\int_s^t\tilde Z_{s,u-}\frac{\lambda\tilde Z^1_{s,u-}}{\lambda\tilde Z^1_{s,u-}+(1-\lambda)\tilde Z^2_{s,u-}}dL^1_u+\int_s^t\tilde Z_{s,u-}\frac{(1-\lambda)\tilde Z^2_{s,u-}}{\lambda\tilde Z^1_{s,u-}+(1-\lambda)\tilde Z^2_{s,u-}}dL^2_u\\&+\int_s^t\int_{\mathbb{R}}\tilde Z_{s,u-}\big\{\frac{\lambda\tilde Z^1_{s,u-}}{\lambda Z^1_{s,u-}+(1-\lambda)\tilde Z^2_{s,u-}}l^1(u,y)+\frac{(1-\lambda)\tilde Z^2_{s,u-}}{\lambda\tilde Z^1_{s,u-}+(1-\lambda)\tilde Z^2_{s,u-}}l^2(u,y)\big\}\tilde N(du,dy).\end{aligned}$$

Thus

$$\begin{aligned}&\tilde\alpha_s(\lambda\tilde Z^1_{s,t}+(1-\lambda)\tilde Z^2_{s,t})\\&=E_{Q^0}\bigg(\int_{]s,t]}(\lambda\tilde Z^1_{s,u-}+(1-\lambda)\tilde Z^2_{s,u-})\xi_u\Big\{\tfrac{\lambda\tilde Z^1_{s,u-}\widetilde{(l^1)}_u}{\lambda\tilde Z^1_{s,u-}+(1-\lambda)\tilde Z^2_{s,u-}}+\tfrac{(1-\lambda)\tilde Z^2_{s,u-}\widetilde{(l^2)}_u}{\lambda\tilde Z^1_{s,u-}+(1-\lambda)\tilde Z^2_{s,u-}}\Big\}^2dA_u\\&\qquad+\int_s^t(\lambda\tilde Z^1_{s,u-}+(1-\lambda)\tilde Z^2_{s,u-})\eta_u d\big\langle\tfrac{\lambda\tilde Z^1_{s,-}}{\lambda\tilde Z^1_{s,-}+(1-\lambda)\tilde Z^2_{s,-}}\cdot L^1+\tfrac{(1-\lambda)\tilde Z^2_{s,-}}{\lambda\tilde Z^1_{s,-}+(1-\lambda)\tilde Z^2_{s,-}}\cdot L^2\big\rangle_u\big|\mathscr{F}_s\bigg)\\&\le E_{Q^0}\bigg(\int_{]s,t]}\xi_u\big[\lambda\tilde Z^1_{s,u-}\widetilde{(l^1)}^2_u+(1-\lambda)\tilde Z^1_{s,u-}\widetilde{(l^2)}^2_u\big]dA_u\\&\qquad+\int_{]s,t]}\eta_u\lambda\tilde Z^1_{s,u-}d\langle L^1\rangle_u+\int_{]s,t]}\eta_u(1-\lambda)\tilde Z^2_{s,u-}d\langle L^2\rangle_u\big|\mathscr{F}_s\bigg)\\&=\lambda\tilde\alpha_s(\tilde Z^1_{s,t})+(1-\lambda)\tilde\alpha_s(\tilde Z^2_{s,t}).\end{aligned}$$

□

Theorem 4.3. *For $\tilde Z\in\mathscr{X}$, let $\tilde\alpha_s(\tilde Z_{s,t})$ be defined by (4.1), then $\tilde C=\big\{\tilde C(B)=(\tilde C_t(B))_{t\in[0,T]},B\in L^\infty(\mathscr{F}_T)\big\}$ defined by*

$$\tilde C_t(B)=\operatorname*{ess\,sup}_{\tilde Z\in\mathscr{X}}\Big\{E_{Q^0}\Big[\tilde Z_{t,T}B\big|\mathscr{F}_t\Big]-\tilde\alpha_t(\tilde Z_{t,T})\Big\}\tag{4.5}$$

is an S-related dynamic convex valuation.

Proof. Similar to Remark 3.9. 4) of Theorem 3.8, one can show that for any $B\in L^\infty(\mathscr{F}_t)$ and $s<t$,

$$\tilde C_s(B)=\operatorname*{ess\,sup}_{\tilde Z\in\mathscr{X}}\big\{E_{Q^0}\big[\tilde Z_{s,t}B\big|\mathscr{F}_s\big]-\tilde\alpha_s(\tilde Z_{s,t})\big\}.$$

Furthermore, from an argument as in the proof of Theorem 3.8, $\tilde C_t(B)$ satisfies the $\mathscr{F}_t$-regularity, (i.e., for any $\mathscr{F}_t$-measurable set D and $B_i\in L^\infty(\mathscr{F}_T)$, $i=1,2$, $\tilde C_t(B_1I_D+B_2I_{D^c})=\tilde C_t(B_1)I_D+$

$\tilde{C}_t(B_2)I_{D^c}$, a.s.) and the S-related property. As for the time-consistency of $\tilde{C}$, for any $s < t$ and $B \in L^\infty(\mathscr{F}_T)$

$$\begin{aligned}\tilde{C}_s(\tilde{C}_t(B)) &= \operatorname*{ess\,sup}_{\tilde{Z}\in\mathscr{Z}} \left\{ E_{Q^0}\left[\tilde{Z}_{s,t}\tilde{C}_t(B)\big|\mathscr{F}_s\right] - \tilde{\alpha}_s(\tilde{Z}_{s,t})\right\} \\ &\geq \operatorname*{ess\,sup}_{\tilde{Z}\in\mathscr{Z}} \left\{ E_{Q^0}\left[\tilde{Z}_{s,t}\tilde{Z}_{t,T}B\big|\mathscr{F}_s\right] - E_{Q^0}\left[\tilde{Z}_{s,t}\tilde{\alpha}_t(\tilde{Z}_{t,T})\big|\mathscr{F}_s\right] - \tilde{\alpha}_s(\tilde{Z}_{s,t})\right\} \\ &= \operatorname*{ess\,sup}_{Z\in\mathscr{Z}} \left\{ E\left[Z_{s,T}B\big|\mathscr{F}_s\right] - \alpha_s(Z_{s,T})\right\} \\ &= C_s(B).\end{aligned} \tag{4.6}$$

On the other hand, $\left\{E_{Q^0}[\tilde{Z}_{t,T}B|\mathscr{F}_t] - \tilde{\alpha}_t(\tilde{Z}_{t,T});\ Z \in \mathscr{Z}\right\}$ is closed under $\vee$, thus there exists a sequence $(\tilde{Z}^n)_{n\in\mathbb{N}} \subset \mathscr{Z}$ such that

$$\tilde{C}_t(B) = \nearrow \lim_{n\to\infty} E_{Q^0}\left[\tilde{Z}^n_{t,T}B\big|\mathscr{F}_t\right] - \tilde{\alpha}_t(\tilde{Z}^n_{t,T})\ ,$$

thus for any $\tilde{Z} \in \mathscr{Z}$, let $\tilde{Z}^{*,n}_u = \tilde{Z}_u I_{u<t} + \tilde{Z}_t\tilde{Z}^n_{t,u}I_{u\geq t}$, then

$$\begin{aligned}&E_{Q^0}\left[\tilde{Z}_{s,t}\tilde{C}_t(B)\big|\mathscr{F}_s\right] - \tilde{\alpha}_s(\tilde{Z}_{s,t}) \\ &\quad = \nearrow \lim_{n\to\infty} E_{Q^0}\left[\tilde{Z}_{s,t}\left\{E_{Q^0}(\tilde{Z}^n_{t,T}B|\mathscr{F}_t) - \tilde{\alpha}_t(\tilde{Z}^n_{t,T})\right\}\big|\mathscr{F}_s\right] - \tilde{\alpha}_s(\tilde{Z}_{s,t}) \\ &\quad = \nearrow \lim_{n\to\infty} E_{Q^0}[\tilde{Z}_{s,t}\tilde{Z}^n_{t,T}B\big|\mathscr{F}_t] - E_{Q^0}\left[\tilde{Z}_{s,t}\tilde{\alpha}_t(\tilde{Z}^n_{t,T})\big|\mathscr{F}_s\right] - \tilde{\alpha}_s(\tilde{Z}_{s,t}) \\ &\quad = \nearrow \lim_{n\to\infty} E_{Q^0}[\tilde{Z}^{*,n}_{s,T}B\big|\mathscr{F}_t] - \tilde{\alpha}_s(\tilde{Z}^{*,n}_{s,T}) \\ &\quad \leq \operatorname{ess\,sup}_{\tilde{Z}\in\mathscr{Z}}\{E_{Q^0}[\tilde{Z}_{s,T}B\big|\mathscr{F}_t] - \tilde{\alpha}_s(\tilde{Z}_{s,T})\} \\ &\quad = \tilde{C}_s(B)\ .\end{aligned}$$

Therefore

$$\tilde{C}_s(\tilde{C}_t(B)) = \operatorname*{ess\,sup}_{\tilde{Z}\in\mathscr{Z}} \left\{E_{Q^0}\left[\tilde{Z}_{s,t}\tilde{C}_t(B)|\mathscr{F}_s\right] - \tilde{\alpha}_s(\tilde{Z}_{s,t})\right\} \leq \tilde{C}_s(B). \tag{4.7}$$

From (4.6) and (4.7), we have

$$\tilde{C}_s(\tilde{C}_t(B)) = \tilde{C}_s(B)$$

and the time consistency follows. □

Similar to Lemma 3.10, we have the following lemma

Lemma 4.4. *For any $B \in L^\infty(\mathscr{F}_T)$, let $\tilde{C}(B)$ be the dynamic convex valuation (DCV) related to S defined by (4.5), then*

1) $\left\{\tilde{Z}_t\big(\tilde{C}_t(B) + \tilde{\alpha}_t(\tilde{Z}_{t,T})\big);\ t \in [0,T]\right\}$ *is an RCLL supermartingale under Q^0 for any $\tilde{Z} \in \mathscr{Z}$;*

2) *if there exists $Z^* \in \mathscr{Z}$ such that $Z^*_t\big(\tilde{C}_t(B)+\tilde{\alpha}_t(Z^*_{t,T})\big)$ is a uniformly integrable martingale under Q^0, then for every $t \in [0,T]$*

$$\tilde{C}_t(B) = E_{Q^0}\left[Z^*_{t,T}B\big|\mathscr{F}_t\right] - \tilde{\alpha}_t(Z^*_{t,T}),\ a.s.$$

To obtain the dynamics of $\tilde C(B)$, for any $\tilde Z_t = Z_0\mathcal{E}\{-\int_0^\cdot \tilde l_u d\widetilde{M}_u + \int_0^\cdot\int_{\mathbb{R}} l(u,y)\tilde N(du,dy) + L\}_t$, we introduce

$$\Lambda_t(\tilde Z) := \int_{]0,t]} \tilde Z_{u-}\xi_u(\tilde l_u)^2 dA_u + \int_{]0,t]} \tilde Z_{u-}\eta_u d\langle L\rangle_u \ .$$

$\Lambda(\tilde Z)$ is a continuous increasing process and the following equality holds

$$\tilde\alpha_s(\tilde Z_{s,t}) = \frac{1}{\tilde Z_s} E_{Q^0}\left[\Lambda_t(\tilde Z) - \Lambda_s(\tilde Z)\big|\mathscr{F}_s\right], \ a.s.. \tag{4.8}$$

Thus we have the following corollary

Corollary 4.5. *For any $B \in L^\infty(\mathscr{F}_T)$, let $\tilde C(B)$ be the S-related DCV related to S defined by* (3.9)*, then*

1′) $\left\{\tilde Z_t \tilde C_t(B) - \Lambda_t(\tilde Z);\ t \in [0,T]\right\}$ *is an RCLL supermartingale under Q^0 for any $\tilde Z \in \mathscr{Z}$;*

2′) *if there exists $Z^* \in \mathscr{Z}$ such that $Z^*_t \tilde C_t(B) - \Lambda_t(Z^*)$ is a uniformly integrable martingale under Q^0, then for every $t \in [0,T]$*

$$\tilde C_t(B) = E_{Q^0}\left[Z^*_{t,T} B\big|\mathscr{F}_t\right] - \tilde\alpha_t(Z^*_{t,T}),\ a.s.$$

Proof. From (4.8), one easily derives

$$\tilde Z_t \tilde C_t(B) - \Lambda_t(\tilde Z) = \tilde Z_t\big(\tilde C_t(B) + \tilde\alpha_t(\tilde Z_{t,T})\big) - E_{Q^0}[\Lambda_T(\tilde Z)\big|\mathscr{F}_t].$$

Since $\{E_{Q^0}[\Lambda_T(\tilde Z)\big|\mathscr{F}_t]\}$ is a uniformly integrable martingale under Q^0, 1′) and 2′) follow from Lemma 4.4. □

For a given $B \in L^\infty(\mathscr{F}_T)$, we consider the following backward semimartingale equation (BSE)

$$\begin{cases} Y_t &= Y_0 - \int_0^t \frac{\xi_u}{2}(\tilde\varphi_u)^2 dA_u - \int_0^t \frac{1}{4\eta_u} d\langle \tilde L\rangle_u + \int_0^t \theta(u) d\widetilde{M}_u + \tilde L_t \\ & \quad + \int_0^t \int_{\mathbb{R}} \frac{\sigma_2(u,y)}{\sigma_1(u)}\{\theta(u) + \sqrt{2}\xi_u\tilde\varphi_u\}\tilde N(du,dy), \quad t < T; \\ Y_T &= B\ . \end{cases} \tag{4.9}$$

The **solution of the BSE (4.9)** is a 4-tuple $(Y, \theta, \varphi, \tilde L)$ satisfying (4.9) such that

(1) θ is a predictable process such that $\theta\cdot\widetilde{M}$ is a BMO-martingale under Q^0. φ is a $\widetilde{\mathscr{P}}$-measurable function with $\varphi(u,y) > -1$ and $\tilde\varphi_u = \int_{\mathbb{R}} \frac{\sigma_2(u,y)}{\sigma_1(u)}\varphi(u,y)K(u;dy)$. Also $\int_0^t\int_{\mathbb{R}}\varphi(u,y)\tilde N(du,dy)$ is a BMO-martingale under Q^0 with $\int_0^T\int_{\mathbb{R}}\varphi(u,y)^2K(u;dy)dA_u \le c < \infty$, Q^0-a.s. for some $c \in \mathbb{R}$;

(2) $\tilde L$ is a BMO martingale under Q^0(i.e., $\tilde L \in BMO(Q^0)$) with $\langle\tilde L\rangle_T \le c < \infty$, Q^0-a.s., which is strongly orthogonal to $\widetilde{M}$ under Q^0;

(3) Y is a bounded RCLL semimartingale.

Remark 4.6.

1. *If S is a continuous semimartingale, i.e., $\sigma_2(u,y)=0$, the BSE equation (4.9) is the same as (4.5) of Mania and Schweizer(2005)*[24].
2. *In general, the BSE (4.9) might not have a solution. However, in many cases, the BSE (4.9) has a solution. Especially, when $B = x + \int_0^T \pi_u dS_u$ for some $\pi \in Adm$,*

$$\begin{cases} \theta(u) &= \sigma_1(u)\pi_u \ , \\ \varphi(u,y) &= 0 \ , \\ \tilde{L}_t &= 0 \ , \\ Y_t &= x + \displaystyle\int_0^t \pi_u dS_u \end{cases}$$

is the solution of the BSE (4.9).

3. *If BSE (4.9) has a solution given by $(Y,\theta,\varphi,\tilde{L})$, one can see that there exists a constant still denoted by c such that*

$$\int_0^T \xi_u(\tilde{\varphi}_u)^2 dA_u + \int_0^T \eta_u d\langle \tilde{L}\rangle_u \le c < \infty, \ Q^0\text{-}a.s., \tag{4.10}$$

since ξ and η are two bounded positive processes.

Theorem 4.7. *If the BSE (4.9) has a solution denoted by $(Y,\theta,\varphi,\tilde{L})$, then for all $t \in [0,T]$*

$$\tilde{C}_t(B) = Y_t, \quad Q^0\text{-}a.s.$$

Proof. For any $\tilde{Z} \in \mathscr{Z}$, there exist a continuous local Q^0-martingale L with $L_0 = 0$ and $\langle L, \widetilde{M}\rangle = 0$, and a functional $l \in \widetilde{\mathscr{P}}$ such that

$$\tilde{Z}_t = 1 - \int_0^t \tilde{Z}_{u-}\tilde{l}_u d\widetilde{M}_u + \int_0^t \int_{\mathbb{R}} \tilde{Z}_{u-} l(u,y)\tilde{N}(du,dy) + \int_0^t \tilde{Z}_{u-} dL_u \ .$$

If BSE (4.9) has a solution $(Y, \theta, \varphi, \tilde{L})$, from Itô's formula we find that

$$\begin{aligned}
Y_t\tilde{Z}_t - \Lambda_t(\tilde{Z}) &= Y_0 Z_0 - \int_0^t Y_{u-}\tilde{Z}_{u-}\tilde{l}_u d\widetilde{M}_u + \int_0^t \int_{\mathbb{R}} Y_{u-}\tilde{Z}_{u-} l(u,y)\tilde{N}(du,dy) + \int_0^t Y_{u-}\tilde{Z}_{u-} dL_u \\
&\quad - \int_0^t \tilde{Z}_{u-}\frac{\xi_u}{2}(\tilde{\varphi}_u)^2 dA_u - \int_0^t \tilde{Z}_{u-}\frac{1}{4\eta_u} d\langle \tilde{L}\rangle_u + \int_0^t \tilde{Z}_{u-}\theta(u) d\widetilde{M}_u + \int_0^t \tilde{Z}_{u-} d\tilde{L}_u \\
&\quad + \int_0^t \int_{\mathbb{R}} \tilde{Z}_{u-}\frac{\sigma_2(u,y)}{\sigma_1(u)}\{\theta(u) + \sqrt{2}\xi_u\tilde{\varphi}_u\}\tilde{N}(du,dy) \\
&\quad - \int_0^t \tilde{Z}_{u-}\tilde{l}_u\theta(u) dA_u + \int_0^t \tilde{Z}_{u-} d\langle \tilde{L}, L\rangle_u \\
&\quad + \int_0^t \int_{\mathbb{R}} \tilde{Z}_{u-} l(u,y)\frac{\sigma_2(u,y)}{\sigma_1(u)}\{\theta(u) + \sqrt{2}\xi_u\tilde{\varphi}_u\}\mu(du,dy) \\
&\quad - \int_0^t \tilde{Z}_{u-}\xi_u(\tilde{l}_u)^2 dA_u - \int_0^t \tilde{Z}_{u-}\eta_u d\langle L\rangle_u \\
&= Y_0 Z_0 + \int_0^t \tilde{Z}_{u-}\{\theta(u) - Y_{u-}\tilde{l}_u\} d\widetilde{M}_u \\
&\quad + \int_0^t \int_{\mathbb{R}} \tilde{Z}_{u-}\Big\{Y_{u-} l(u,y) + \frac{\sigma_2(u,y)}{\sigma_1(u)}\{1 + l(u,y)\}\{\theta(u) + \sqrt{2}\xi_u\tilde{\varphi}_u\}\Big\}\tilde{N}(du,dy) \\
&\quad + \int_0^t Y_{u-}\tilde{Z}_{u-} dL_u + \int_0^t \tilde{Z}_{u-} d\tilde{L}_u \\
&\quad - \int_0^t \tilde{Z}_{u-}\frac{\xi_u}{2}\Big\{(\tilde{\varphi}_u)^2 - 2\sqrt{2}\tilde{\varphi}_u\tilde{l}_u + 2(\tilde{l}_u)^2\Big\} dA_u \\
&\quad - \int_0^t \tilde{Z}_{u-}\frac{1}{4\eta_u} d\langle \tilde{L}\rangle_u + \int_0^t \tilde{Z}_{u-} d\langle \tilde{L}, L\rangle_u - \int_0^t \tilde{Z}_{u-}\eta_u d\langle L\rangle_u\ .
\end{aligned}$$

As ξ and η are positive, bounded and predictable processes, $\{Y_t\tilde{Z}_t - \Lambda_t(\tilde{Z}); t \in [0,T]\}$ is a local Q^0-supermartingale. Furthermore, as Y is a bounded process, $\tilde{Z}$ is a uniformly integrable Q^0-martingale and $\Lambda_t(\tilde{Z}) \le \Lambda_T(\tilde{Z})$ and $E_{Q^0}[\Lambda_T(\tilde{Z})] = \tilde{\alpha}_{0,T}(\tilde{Z}) < \infty$ for $\tilde{Z} \in \mathscr{Z}$, so one sees that $\{Y_t\tilde{Z}_t - \Lambda_t(\tilde{Z}); t \in [0,T]\}$ is uniformly integrable under Q^0. Hence, $\{Y_t\tilde{Z}_t - \Lambda_t(\tilde{Z}); t \in [0,T]\}$ is a true Q^0-supermartingale for any $\tilde{Z} \in \mathscr{Z}$. Therefore, for any $\tilde{Z} \in \mathscr{Z}$

$$Y_t\tilde{Z}_t - \Lambda_t(\tilde{Z}) \ge E_{Q^0}\big[B\tilde{Z}_T - \Lambda_T(\tilde{Z})\big|\mathscr{F}_t\big],$$

thus

$$Y_t \ge E_{Q^0}\big[\tilde{Z}_{t,T}B\big|\mathscr{F}_t\big] - \tilde{\alpha}_t(\tilde{Z}_{t,T})$$

and hence

$$Y_t \ge \operatorname*{ess\,sup}_{\tilde{Z}\in\mathscr{Z}}\Big\{E_{Q^0}\big[\tilde{Z}_{t,T}B\big|\mathscr{F}_t\big] - \tilde{\alpha}_t(\tilde{Z}_{t,T})\Big\} = \tilde{C}_t(B), \quad \text{a.s.}$$

Furthermore, if we let

$$Z_t^* := \mathcal{E}\big\{-\int_0^\cdot \frac{1}{\sqrt{2}}\tilde{\varphi}_u d\widetilde{M}_u + \int_0^\cdot \int_{\mathbb{R}} \frac{1}{\sqrt{2}}\varphi(u,y)\tilde{N}(du,dy) + \int_0^\cdot \frac{1}{2\eta_u} d\tilde{L}_u\big\}_t,$$

it follows from (4.13) that

$$\begin{aligned}
\tilde{\alpha}_0(\tilde{Z}_{0,T}) &= E_{Q^0}\big(\textstyle\int_0^T \tilde{Z}_{0,u-}\xi_u(\tilde{l}_u)^2 dA_u + \int_0^T \tilde{Z}_{0,u-}\eta_u d\langle L\rangle_u\big) \\
&= E_{Q^0}\big[\tilde{Z}_{0,T}\{\textstyle\int_0^T \xi_u(\tilde{l}_u)^2 dA_u + \int_0^T \eta_u d\langle L\rangle_u\}\big] \\
&\le c < \infty,
\end{aligned}$$

for some constant $c > 0$, thus $Z^* \in \mathscr{Z}$ and $\{Y_t Z_t^* - \Lambda_t(\tilde Z^*); t \in [0,T]\}$ is a uniformly integrable Q^0-martingale. Thus

$$Y_t Z_t^* - \Lambda_t(Z^*) = E_{Q^0}\big[Z_T^* B - \Lambda_T(Z^*)\big|\mathscr{F}_t\big],$$

which implies

$$\begin{aligned} Y_t &= E_{Q^0}\big[Z_{t,T}^* B\big|\mathscr{F}_t\big] - \tilde\alpha_t(Z_{t,T}^*) \\ &\le \operatorname{ess\,sup}_{\tilde Z \in \mathscr{Z}} \Big\{E_{Q^0}\big[\tilde Z_{t,T} B\big|\mathscr{F}_t\big] - \tilde\alpha_t(\tilde Z_{t,T})\Big\} \\ &= \tilde C_t(B). \end{aligned}$$

So finally we get the desired result

$$\tilde C_t(B) = Y_t \ a.s.$$

for all $t \in [0,T]$, which completes the proof. □

4.2 The S-related DCV generated by $\hat\alpha$

We now consider the S-related dynamic convex valuation generated by $\hat\alpha$. Let

$$\hat{\mathscr{Z}} := \big\{\tilde Z \in \tilde{\mathbb{Z}}_e;\ \hat\alpha_0(\tilde Z_{0,T}) < \infty\big\}.$$

By similar arguments as in the preceding case, one can show that for each $\tilde Z \in \hat{\mathscr{Z}}$ and for any $s < t$

(i") $\displaystyle \operatorname*{ess\,inf}_{\tilde Z \in \hat{\mathscr{Z}}} \Big\{\hat\alpha_s(\tilde Z_{s,t})\Big\} = 0;$

(ii") $\hat\alpha_s(\tilde Z_{s,T}) = \hat\alpha_s(\tilde Z_{s,t}) + E_{Q^0}[\tilde Z_{s,t}\hat\alpha_t(\tilde Z_{t,T})\big|\mathscr{F}_s]$, Q^0-a.s.;

(iii") for any $\tilde Z^i \in \hat{\mathscr{Z}}$, $i = 1,2$ and a $\mathscr{F}_t$-measurable set D, let $\tilde Z_u = \tilde Z_u^1 I_{u<t} + \tilde Z_t^1\{\tilde Z_{t,u}^1 I_D + \tilde Z_{t,u}^2 I_{D^c}\} I_{u\ge t}$, then $\hat\alpha_t(\tilde Z_{t,T}) = \hat\alpha_t(\tilde Z_{t,T}^1) I_D + \hat\alpha_t(\tilde Z_{t,T}^2) I_{D^c}$;

(iv") $\displaystyle \lim_{h\searrow 0} \hat\alpha_t(\tilde Z_{t,t+h}) = 0$, Q^0-a.s.

Also for any $\tilde Z^i \in \hat{\mathscr{Z}}$, $i = 1,2$ and $\lambda \in [0,1]$,

$$\hat\alpha_s\left(\lambda \tilde Z_{s,t}^1 + (1-\lambda)\tilde Z_{s,t}^2\right) \le \lambda\hat\alpha_s(\tilde Z_{s,t}^1) + (1-\lambda)\hat\alpha_s(\tilde Z_{s,t}^2), \quad Q^0\text{-a.s.}$$

Adapting the proof of Theorem 4.3, one directly derives

Theorem 4.8. *For $\tilde Z \in \hat{\mathscr{Z}}$, let $\hat\alpha_s(\tilde Z_{s,t})$ be defined by (4.2), then $\hat C = \big\{\hat C(B) = (\hat C_t(B))_{t\in[0,T]}, B \in L^\infty(\mathscr{F}_T)\big\}$ defined by*

$$\hat C_t(B) = \operatorname*{ess\,sup}_{\tilde Z \in \hat{\mathscr{Z}}} \Big\{E_{Q^0}\Big[\tilde Z_{t,T} B\Big|\mathscr{F}_t\Big] - \hat\alpha_t(\tilde Z_{t,T})\Big\} \tag{4.11}$$

is an S-related dynamic convex valuation (DCV).

For any $\tilde{Z}_t = Z_0\mathcal{E}\{ -\int_0^{\cdot} \tilde{l}_u d\widetilde{M}_u + \int_0^{\cdot}\int_{\mathbb{R}} l(u,y)\tilde{N}(du,dy) + L\}_t \in \hat{\mathscr{Z}}$, we introduce

$$\hat{\Lambda}_t(\tilde{Z}) := \int_0^t\int_{\mathbb{R}} \tilde{Z}_{u-}\xi(u,y)l(u,y)^2K(u;dy)dA_u + \int_{]0,t]} \tilde{Z}_{u-}\eta_u d\langle L\rangle_u \,.$$

Similar to Corollary 4.5, we have the following dynamic principle

Corollary 4.9. *For any $B \in L^\infty(\mathscr{F}_T)$, let $\hat{C}(B)$ be the S-related dynamic convex valuation defined by* (4.11)*, then*

1″) $\left\{\tilde{Z}_t\hat{C}_t(B) - \hat{\Lambda}_t(\tilde{Z});\ t\in[0,T]\right\}$ *is a RCLL supermartingale under Q^0 for any $\tilde{Z}\in\hat{\mathscr{Z}}$;*

2″) *if there exists $Z^*\in\hat{\mathscr{Z}}$ such that $Z^*_t\hat{C}_t(B) - \hat{\Lambda}_t(Z^*)$ is a uniformly integrable martingale under Q^0, then for every $t\in[0,T]$*

$$\hat{C}_t(B) = E_{Q^0}\left[Z^*_{t,T}B\middle|\mathscr{F}_t\right] - \hat{\alpha}_t(Z^*_{t,T}),\ a.s.$$

We consider the following backward semimartingale equation

$$\begin{cases} \hat{Y}_t &= \hat{Y}_0 - \displaystyle\int_0^t\int_{\mathbb{R}} \xi(u,y)\varphi(u,y)^2K(u;dy)dA_u - \int_0^t \frac{1}{4\eta_u}d\langle\hat{L}\rangle_u \\ &\quad + \displaystyle\int_0^t \theta(u)d\widetilde{M}_u + \int_0^t\int_{\mathbb{R}}\left\{\frac{\sigma_2(u,y)}{\sigma(u)}\theta(u) + 2\xi(u,y)\varphi(u,y)\right\}\tilde{N}(du,dy) + \hat{L}_t\,, \\ \hat{Y}_T &= B \end{cases} \tag{4.12}$$

The **solution of the BSE (4.12)** is a 4-tuple $(\hat{Y},\theta,\varphi,\hat{L})$ satisfying (4.9) such that

(1) θ is a predictable process such that $\theta\cdot\widetilde{M}$ is a BMO-martingale under Q^0. φ is a $\widetilde{\mathscr{P}}$-measurable function with $\varphi(u,y) > -1$ such that $\int_0^t\int_{\mathbb{R}}\varphi(u,y)\tilde{N}(du,dy)$ is a BMO under Q^0 with $\int_0^T\int_{\mathbb{R}}\varphi(u,y)^2K(u;dy)dA_u \le c < \infty$, Q^0-a.s. for some $c\in\mathbb{R}$;

(2) $\hat{L}$ is a BMO martingale under Q^0(i.e., $\hat{L}\in BMO(Q^0)$) with $\langle\hat{L}\rangle_T \le c < \infty$, Q^0-a.s., which is strongly orthogonal to $\widetilde{M}$ under Q^0;

(3) Y is a bounded RCLL semimartingale.

Remark 4.10.

1. *Even if S is a continuous semimartingale, i.e., $\sigma_2(u,y)=0$, the BSE equation (4.12) is quite different from (4.5) or (4.9) of Mania and Schweizer(2005)[24], therefore the S-related DCV generated by $\hat{\alpha}$ is totally different from the* 'dynamic exponential utility indifference valuation' *(see [24]). We think that the wealth related valuation is more realistic than the one related to a utility function and the more or less artificial corresponding indifference price.*

2. *In general, the BSE (4.12) may not have a solution. However, in many cases, the BSE (4.12) has a solution. Especially, when* $B = x + \int_0^T \pi_u dS_u$ *for some* $\pi \in Adm$,

$$\begin{cases} \theta(u) & = \sigma_1(u)\pi_u \ , \\ \varphi(u,y) & = 0 \ , \\ \tilde{L}_t & = 0 \ , \\ \hat{Y}_t & = x + \displaystyle\int_0^t \pi_u dS_u \end{cases}$$

is also the solution of the BSE (4.12).

3. *If the BSE (4.12) has a solution given by* $(\hat{Y}, \theta, \varphi, \hat{L})$, *one can see that there exists a constant still denoted by c such that*

$$\int_0^T \int_{\mathbb{R}} \xi(u,y)\varphi(u,y)^2 K(u;dy) dA_u + \int_0^T \eta_u d\langle \tilde{L} \rangle_u \leq c < \infty, \ Q^0\text{-}a.s., \tag{4.13}$$

since ξ *and* η *are two bounded positive process.*

Theorem 4.11. *If the BSE (4.9) has a solution denoted by* $(\hat{Y}, \theta, \varphi, \hat{L})$, *then for all* $t \in [0, T]$

$$\hat{C}_t(B) = \hat{Y}_t, \quad Q^0\text{-}a.s.$$

Proof. Recall that $\tilde{l}_u = \int_{\mathbb{R}} \frac{\sigma_2(u,y)}{\sigma_1(u)} l(u,y)K(u;dy)$, so it follows from Itô's formula that

$$
\begin{aligned}
Y_t\tilde{Z}_t - \hat{\Lambda}_t(\tilde{Z}) &= Y_0 Z_0 + \int_0^t \tilde{Z}_{u-}\{\theta(u) - Y_{u-}\tilde{l}_u\} d\widetilde{M}_u \\
&+ \int_0^t \int_{\mathbb{R}} \tilde{Z}_{u-}\left\{ Y_{u-} l(u,y) + \left\{ \frac{\sigma_2(u,y)}{\sigma(u)}\theta(u) + 2\xi(u,y)\varphi(u,y) \right\} \{1 + l(u,y)\} \right\} \tilde{N}(du,dy) \\
&+ \int_0^t Y_{u-}\tilde{Z}_{u-} dL_u + \int_0^t \tilde{Z}_{u-} d\hat{L}_u \\
&- \int_0^t\int_{\mathbb{R}} \tilde{Z}_{u-}\xi(u,y)\varphi(u,y)^2 K(u;dy)dA_u - \int_0^t \tilde{Z}_{u-}\tilde{l}_u\theta(u)dA_u \\
&+ \int_0^t \int_{\mathbb{R}} \tilde{Z}_{u-}\left\{ \frac{\sigma_2(u,y)}{\sigma(u)}\theta(u) + 2\xi(u,y)\varphi(u,y) \right\} l(u,y)\tilde{K}(u;dy)dA_u \\
&- \int_0^t\int_{\mathbb{R}} \tilde{Z}_{u-}\xi(u,y)l(u,y)^2 K(u;dy)dA_u \\
&- \int_0^t \tilde{Z}_{u-}\frac{1}{4\eta_u} d\langle \hat{L}\rangle_u + \int_0^t \tilde{Z}_{u-} d\langle L, \hat{L}\rangle_u - \int_{]0,t]} \tilde{Z}_{u-}\eta_u d\langle L\rangle_u \\
&= Y_0 Z_0 + \int_0^t \tilde{Z}_{u-}\{\theta_1(u) - Y_{u-}\tilde{l}_u\} d\widetilde{M}_u \\
&+ \int_0^t \int_{\mathbb{R}} \tilde{Z}_{u-}\left\{ Y_{u-} l(u,y) + \left\{ \frac{\sigma_2(u,y)}{\sigma(u)}\theta(u) + 2\xi(u,y)\varphi(u,y) \right\} \{1 + l(u,y)\} \right\} \tilde{N}(du,dy) \\
&+ \int_0^t Y_{u-}\tilde{Z}_{u-} dL_u + \int_0^t \tilde{Z}_{u-} d\hat{L}_u \\
&- \int_0^t \int_{\mathbb{R}} \tilde{Z}_{u-}\xi(u,y)\{l(u,y) - \varphi(u,y)\}^2 K(u;dy)dA_u \\
&- \int_0^t \tilde{Z}_{u-}\frac{1}{4\eta_u} d\langle \hat{L}\rangle_u + \int_0^t \tilde{Z}_{u-} d\langle L, \hat{L}\rangle_u - \int_{]0,t]} \tilde{Z}_{u-}\eta_u d\langle L\rangle_u .
\end{aligned}
$$

The rest is the same as the proof of Theorem 4.7. □

In the above results we present a to our knowledge new valuation which relates the value of general claims directly to accessible wealths. In this way we avoid to use controversially discussed pricing rules like e.g. utility indifference. This new valuation has all desirable properties and even holds in the general jump market given above. Accessibility of this valuation is guaranteed by the representation as BSE. It would be interesting to put different valuations to a benchmark test. This however goes far beyond the aim of this research and far beyond our numerical capabilities.

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