Tamm's theorem for log-analytic functions

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Important o-minimal structures on the real field

- $\mathbb{R}$: The pure real field.
- $\mathbb{R}_{an}$: The real field augmented by restricted analytic functions. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called restricted analytic if it is of the form

$$f(x) := \begin{cases} p(x), \text{ if } x \in [-1,1]^n, \\ 0 \text{ else.} \end{cases},$$

  where $p(x)$ is a power series which converges on a neighbourhood of $[-1,1]^n$. The definable sets and functions are exactly the globally subanalytic ones.
- $\mathbb{R}_{an,exp}$: The structure $\mathbb{R}_{an}$ augmented by the exponential function exp.

A parametric result of Tamm's theorem for $\mathbb{R}_{\text{an}}$

L. van den Dries and C. Miller have shown the following theorem:

Theorem (A parametric version of Tamm's theorem)

- *Let $f : \mathbb{R}^{n+m} \to \mathbb{R}$ be definable in $\mathbb{R}_{\text{an}}$. Then there exists $N \in \mathbb{N}$ such that for all $(x_0, y_0) \in \mathbb{R}^{n+m}$, if $y \mapsto f(x_0, y)$ is C^N in a neighbourhood of y_0, then $y \mapsto f(x_0, y)$ is real analytic in a neighbourhood of y_0.*
- *The set* $\{(x, y) \in \mathbb{R}^{n+m} \mid f(x, -) \text{ is real analytic at } y\}$ *is definable in* $\mathbb{R}_{\text{an}}$.

Question: Do this two parts of the theorem hold in $\mathbb{R}_{\text{an,exp}}$?
$\Rightarrow$ In general not.

Counterexample to the first claim in $\mathbb{R}_{\text{an,exp}}$.

- Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) := \begin{cases} xe^{-\frac{1}{y^2}}, \text{ if } y > 0, \\ 0 \text{ else.} \end{cases}$$

- For $x \neq 0$, $f(x,-)$ is C^∞, but not real analytic at $y = 0$.

Counterexample to the second claim in $\mathbb{R}_{\text{an,exp}}$

- Consider the function

$$f(x,y) := \begin{cases} |y|^{\frac{1}{x}}, \text{ if } x > 0 \text{ and } y \neq 0, \\ 0 \text{ else.} \end{cases}$$

and the set

$$A := \{(x,y) \in \mathbb{R}^2 \mid f(x,-) \text{ is real analytic at } y\}.$$

- We see that

$$M := \{x \in \mathbb{R} \mid f(x,-) \text{ is real analytic at } y = 0\}$$

is not definable in $\mathbb{R}_{\text{an,exp}}$. So A isn't definable in $\mathbb{R}_{\text{an,exp}}$ as well.

Remark

Tamm's theorem doesn't hold for C^∞ instead of real analytic in general.

Main Question: Is there a natural class of functions which are definable in the structure $\mathbb{R}_{\text{an,exp}}$ such that the parametric version of Tamm's theorem holds?

Log-analytic functions

Let $X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}$ be definable in $\mathbb{R}_{\text{an,exp}}$.

Definition (log-analytic functions)

- f is log-analytic of type 0, if f is the restriction of a globally subanalytic function on X.
- We call a function $f : X \to \mathbb{R}$ log-analytic of type $r \in \mathbb{N}$, if there is a decomposition $\mathcal{C}$ of X in analytic cells definable in $\mathbb{R}_{\text{an,exp}}$ such that for all $C \in \mathcal{C}$

$$f(x) = F(g_1(x), ..., g_l(x), \log(g_{l+1}(x)), ..., \log(g_m(x))),$$

where F is globally subanalytic and $g_1, ..., g_m : C \to \mathbb{R}$ are log-analytic functions of type less than r. There is a $i \in \{l+1, ..., m\}$ such that g_i is log-analytic of type $r-1$.

Main theorem

Main theorem

- Let $f : \mathbb{R}^{n+m} \to \mathbb{R}$ be a log-analytic function. Then there exists $N \in \mathbb{N}$ such that for all $(x_0, y_0) \in \mathbb{R}^{n+m}$ if $y \mapsto f(x_0, y)$ is C^N in a neighbourhood of y_0 then $y \mapsto f(x_0, y)$ is real analytic in a neighbourhood of y_0.
- The set $M := \{(x, y) \in \mathbb{R}^{n+m} \mid f(x, -) \text{ is real analytic } \}$ is definable in $\mathbb{R}_{\text{an,exp}}$.

Convention

"Definable" means definable in $\mathbb{R}_{\text{an,exp}}$.

Proof of the "Main theorem"

Theorem (Lion/Rolin preparation for log-analytic functions)

Let $X \subseteq \mathbb{R}^{n+1}$ definable in $\mathbb{R}_{\text{an,exp}}$ and $f : X \to \mathbb{R}$ a log-analytic function. Then there is a $r \in \mathbb{N}$ and a decomposition $\mathcal{C}$ of X into definable analytic cells such that for all $C \in \mathcal{C}$

$$f(x, y) = A(x){y_0}^{q_0} \cdot ... \cdot {y_r}^{q_r} U(x, y_0, ..., y_r),$$

where

$$y_0 = |y - \Theta_0(x)|, y_1 = |\log(y_0) - \Theta_1(x)|, ..., y_r = |\log(y_{r-1}) - \Theta_r(x)|$$

such that $A, \Theta_0, ..., \Theta_r$ are log-analytic functions on the base of C, U is a special unit in $y_0, ..., y_r$, $q_i \in \mathbb{Q}$ and $\Theta_i \equiv 0$ or $y_i \leq M|\Theta_i|$ for all i and a $M \in \mathbb{R}$.

Proof steps of the "Main Theorem"

Conclusion

Let $f : A \times (0,1) \to \mathbb{R}$ a log-analytic function with $A \subseteq \mathbb{R}^m$. There are definable sets $A_1, ..., A_m$ with $A = \bigcup A_i$, definable functions $h_1, ..., h_m$ with $h_i : A_i \to (0,1)$ and log-analytic functions $f_1, .., f_m$ such that f_i is prepared and $f = f_i$ holds on $(0, h_i)$. Here

$$(0, h_i) := \{(x, y) \mid x \in A_i, 0 < y < h_i(x)\} \text{ for } i \in \{1, ..., M\}.$$

Proof of the "Main theorem"

Reduction of real analyticity to dimension one: Let $U \subseteq \mathbb{R}^n$ and $f : U \to \mathbb{R}$ be a log-analytic function. Let $x \in U$ and $y \in \mathbb{R}^n$. We consider the function $t \mapsto f(x + yt)$ in a small intervall around zero.

- We call f G^k at x, if $t \mapsto f(x + yt)$ is C^k at $t = 0$ for all $y \in \mathbb{R}^n$ and $y \mapsto \frac{d^k f(x+yt)}{dt^k}(0) : \mathbb{R}^n \to \mathbb{R}$ is given by a homogeneous polynomial in y of degree k.
- If f is G^k for all $k \in \mathbb{N}$, then f is called G^∞.

Lemma (van den Dries/Miller)

f is real analytic at x if and only if f is G^∞ at x and there exists $\epsilon > 0$ such that for all $y \in \mathbb{R}^n$ with $|y| < 1$ the function $t \mapsto f(x + yt)$ is real analytic on $(-\epsilon, \epsilon)$.

Proof of the "Main theorem"

Definition (Flatness)

Let $U \subseteq \mathbb{R}^n$ be open and $a \in U$. Let $f : U \to \mathbb{R}$ be a function. We call f N-flat at a if f is C^N at a and all partial derivatives of f of order less than N vanish at a. We call f flat at a if all partial derivatives vanish at a.

Lemma

Let $A \subseteq \mathbb{R}^n$ definable and $f : A \to \mathbb{R}$ be a log-analytic function. Then there exists $N \in \mathbb{N}$ such that for all $(x, y) \in A$ with $y \in \text{int}(A_x)$ the following holds:
f is N-flat at a $\Rightarrow f(x, -) \equiv 0$ in a neighbourhood of y on A_x.

Proof of the "Main theorem"

Conclusions

Let $U \subseteq \mathbb{R}^n$ be open and definable connected and $(f_i)_{i\in\mathbb{N}}$ C^∞ fucktions which are log-analytic on U. Then the following holds:

(1) f_i is flat at $a_0 \in U \Rightarrow f_i \equiv 0$ on U.

(2) Let $Z(f_i) := \{x \in U \mid f_i(x) = 0\}$. Then there exists $M \in \mathbb{N}$, such that $\bigcap_{i\in\mathbb{N}} Z(f_i) = \bigcap_{i\leq M} Z(f_i)$ holds.

- Reduce the property G^k on the zero set of a definable function: $f(x, -)$ is G^k at y if and only if there is a certain definable C^∞ log-analytic function $w_k : \mathbb{R}^{n+m} \times \mathbb{R}^n \to \mathbb{R}$ with $w_k(x, y, z) = 0$ for all $z \in \mathbb{R}^n$.
- There exists $N \in \mathbb{N}$ such that for all $(x, y) \in \mathbb{R}^{n+m}$ the log-analytic function $f(x, -)$ is G^N at y if and only if $f(x, -)$ is G^∞ at y.

Proof of the "Main theorem"

Main step: Consider the definable function $F : \mathbb{R}^{m+n} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$,

$$F(x, y, z, t) := f(x, y + tz)$$

Set $v := (x, y, z)$.

Lemma

There is a $N \in \mathbb{N}$, so that for all $v \in \mathbb{R}^{n+m} \times \mathbb{R}^n$ holds: $F(v, -)$ is C^N at $t = 0$ if and only if $F(v, -)$ is real analytic at $t = 0$.

Proof of the "Main theorem"

Sketch of proof:

- Apply the Lion/Rolin preparation theorem on F in the variable t:

$$F(v,-) := A(v)t_0^{q_0} \cdot \dots \cdot t_r^{q_r} U(v, t_0, \dots, t_r)$$

- We get a multidimensional Puiseux-series in the Variables $t_1, \dots, t_r$.
- Big step: We can find a function g, which is real analytic at $t = 0$, so that $F - g$ is $N-$flat for all $N \in \mathbb{N}$. $\Rightarrow F - g \equiv 0$ at a neighbourhood of 0. $\Rightarrow F(v,-)$ is real analytic at $t = 0$.