
Real Algebraic Geometry I – Exercise Sheet 10

Exercise 1 (4P). Adapt the proof of Cassels' Theorem 2.1.2 to show that an integer that is a sum of 3 squares of rational numbers is also a sum of 3 squares of integers.

Exercise 2 (4P). Let K be a field with $\text{char } K \neq 2$, $f \in K[\underline{X}] \cap \sum K(\underline{X})^2$. Show that $f(a) \in \sum K^2$ for all $a \in K^n$.

Exercise 3 (4P). Which polytopes arise as the Newton polytope of a real polynomial of degree 8 in two variables that is a sum of fourth powers of polynomials?

Exercise 4 (3P). Let $f = X^2Y^2 + X^2Z^2 + Y^2Z^2 - 4XYZ + W^4 \in \mathbb{R}[W, X, Y, Z]$. Show that f is nonnegative on \mathbb{R}^4 while it is not a sum of squares in $\mathbb{R}[W, X, Y, Z]$.

Exercise 5 (3P). Let K be a Euclidean field. Write

$$f := 2X_1^4 - 12X_1^3X_2 + 30X_1^2X_2^2 - 36X_1X_2^3 + 17X_2^4 \in K[X_1, X_2]$$

as a sum of squares of polynomials.

Please submit until Thursday, January 19, 2017, 11:44 in the box named RAG I, Number 10, near to the room F411.