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Real Algebraic Geometry I – Exercise Sheet 12

Exercise 1 (4P). Let A be a commutative ring and $P, Q \in \operatorname{sper} A$. Show that the following are equivalent:

(a) There does not exist a prime cone H in A with $P \subseteq H$ and $Q \subseteq H$.

(b) There is $f \in A$ with $f(P) > 1$ and $f(Q) < 0$.

Exercise 2 (4P). Let R be a real closed field and $p \in R[\underline{X}]$. Show that the following are equivalent:

(a) $p \geq 0$ on R^n

(b) $\widehat{p} \geq 0$ on $\operatorname{sper} R[\underline{X}]$

Hint: Consider R as a ordered subfield of all representations fields R_P of prime cones P of A.

Exercise 3 (4P). Let A and B be commutative rings, $\varphi\colon A \to B$ a ring homomorphism and P a prime cone of A. Show that the following are equivalent:

(a) There exists a prime cone Q of B with $\varphi^{-1}(Q) = P$.

(b) For all $r \in \mathbb{N}$, all $a, a_1, ..., a_r \in P$ with $a \notin -P$ and all $b_1, ..., b_r \in B$

$$\varphi(a) + \sum_{i=1}^{r} \varphi(a_i) b^2 \neq 0.$$

Exercise 4 (4P). Let A be a commutative ring and $P, Q_1, Q_2 \in \operatorname{sper}(A)$ with respective supports $\mathfrak{p}, \mathfrak{q}_1, \mathfrak{q}_2$. Prove:

(a) $P \subseteq Q_1 \cup Q_2$ implies that there is an i such that $\mathfrak{p} \subseteq \mathfrak{q}_i$.

(b) $P = Q_1 \cup Q_2$ implies that there is an i such that $P = Q_i$.

Exercise 5 (4P). Let A be a commutative ring and P a prime cone of A. Show that the following are equivalent:

(a) P is a minimal element of $\operatorname{sper} A$ (partially ordered by inclusion).

(b) $\forall a \in \operatorname{supp}(P) : \exists k \in \mathbb{N}_0 : \exists b \in P \setminus -P : \exists c \in \sum (P \setminus -P) A^2 : a^{2k} b + c = 0$

Please submit until Thursday, February 2, 2017, 11:44 in the box named RAG I, Number 10, near to the room F411.