Universität Konstanz
Department of Mathematics and Statistics
Winter Term 2016/2017

Tom-Lukas Kriel
María López Quijorna
Markus Schweighofer

Real Algebraic Geometry I – Exercise Sheet 2

Exercise 1 (4P).

(a) Let A be a commutative ring, $m \in \mathbb{N}_0$ and $g_1, ..., g_m \in A$. Show that the set

$$T := \sum_{\alpha \in \{0,1\}^m} \left(\sum A^2 \prod_{i=1}^{m} g_i^{\alpha_i} \right)$$

is the smallest preorder of A containing $g_1, ..., g_m$. We say that T is finitely generated (generated by $g_1, ..., g_m$).

(b) Let $T \subseteq \mathbb{R}[X_1, ..., X_n]$ be a finitely generated preorder and $I \subseteq \mathbb{R}[X_1, \ldots, X_n]$ an ideal. Show that $T + I$ is also a finitely generated preorder.

Exercise 2 (6P).

(a) Let $s \in \sum \mathbb{R}[X]^2$ and $a \in \mathbb{R}$ with $s(a) = 0$. Show that $(X - a)^2 \mid s$ in $\mathbb{R}[X]$.

(b) Let $f, g \in \mathbb{R}[X]$ be polynomials of degree 1. Analyze under which circumstances the set

$$\sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 f + \sum \mathbb{R}[X]^2 g$$

is a preorder.

Hint for (b): Try to reduce the question to finitely many cases. This can be done by looking at a change of coordinates induced by an appropriate ring automorphism of $\mathbb{R}[X]$.

Exercise 3 (6P). Let A be the ring of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Answer the following questions:

(a) Is any continuous function from $\mathbb{R}$ to $\mathbb{R}_{\geq 0}$ a square in the ring A?

(b) Is there an embedding of ordered sets from $\{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ (with its natural ordering) into the set of all preorders of A (ordered by inclusion)?

(c) Is there a preorder P of A with $P \cap -P = \{0\}$ and $P \cup -P = A$?

Please submit until Thursday, November 10, 2016, 11:44 in the box named RAG I, Number 10, near to the room F411.