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Real Algebraic Geometry II – Exercise Sheet 2

Exercise 1 (5P) Suppose M is a set, define $\mathscr{F}_0 := \{U \in \mathscr{P}(M) \mid M \setminus U \text{ is finite}\}$. Show the following:

(a) Let $\mathscr{F}$ be an ultrafilter on M. Then either $\mathscr{F}_0 \subseteq \mathscr{F}$ or there exists an $s \in M$ such that $\mathscr{F} = \{U \in \mathscr{P}(M) \mid s \in U\}$. An ultrafilter of the $\left\{\begin{matrix}\text{first}\\ \text{second}\end{matrix}\right\}$ type is called $\left\{\begin{matrix}\text{free}\\ \text{principal}\end{matrix}\right\}$.

(b) Show that M is finite if and only if every ultrafilter on M is principal.

(c) Determine $\{\bigcap \mathscr{F} \mid \mathscr{F} \text{ ultrafilter on } M\}$.

Exercise 2 (5P) Let M be a set.

(a) Find a condition that characterizes when a subset of $\mathscr{P}(M)$ generates a filter on M (in the sense that there is a smallest filter on M containing it).

(b) Show that a countably infinite subset of $\mathscr{P}(M)$ never generates a free ultrafilter on M.

(c) Show that every filter on M is an intersection of ultrafilters.

Exercise 3 (14P) Let I be a set, $(K_i, \leq_i)_{i \in I}$ a family of ordered fields and $\mathscr{U}$ an ultrafilter on I.

(a) Show that

$$\mathfrak{m} := \left\{(a_i)_{i \in I} \in \prod_{i \in I} K_i \mid \{i \in I \mid a_i = 0\} \in \mathscr{U}\right\}$$

is a maximal ideal of the ring $\prod_{i \in I} K_i$ so that

$$R := \left(\prod_{i \in I} K_i\right) \Big/ \mathscr{U} := \left(\prod_{i \in I} K_i\right) \Big/ \mathfrak{m}$$

is a field.

(b) Show that

$$\overline{(a_i)_{i\in I}}^{\mathrm{m}} \leq \overline{(b_i)_{i\in I}}^{\mathrm{m}} \;:\Longleftrightarrow\; \{i \in I \mid a_i \leq b_i\} \in \mathscr{U} \qquad \left((a_i)_{i\in I}, (b_i)_{i\in I} \in \prod_{i\in I} K_i\right)$$

defines an order $\leq$ of the field R so that

$$\left(\prod_{i\in I}(K_i, \leq_i)\right) \Big/ \mathscr{U} := (R, \leq)$$

is an ordered field. We call this ordered field the *ultraproduct* of the ordered fields $(K_i, \leq_i)$, $i \in I$, along the ultrafilter $\mathscr{U}$.

(c) Show that R is Euclidean if K_i is Euclidean for each $i \in I$.

(d) Show that R is real closed if K_i is real closed for each $i \in I$.

Now let $\mathscr{U}$ be a free ultrafilter on $I := \mathbb{N}$.

(e) Show that $(R, \leq)$ is not Archimedean.

(f) Show that every convergent [$\rightarrow$ 1.1.9(b)] sequence $(a_n)_{n\in\mathbb{N}}$ in R is eventually constant.

(g) Endow R with the topology induced by the order $\leq$ in the sense that it is generated by $\{\{x \in R \mid a < x\} \mid a \in R\} \cup \{\{x \in R \mid x < a\} \mid a \in R\}$. Show that 1 is then in the closure of $I := \{x \in R \mid 0 \leq x < 1\}$ but it is not the limit of any sequence in I. This gives the counterexample that was promised in Exercise 4 of Sheet 1.

Please submit until Tuesday, May 9, 2017, 11:44 in the box named RAG II near to the room F411.