The real closure of a Hardy field

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If f is a differentiable function from some half-line $(a, +\infty)$ to $\mathbb{C}$, we will denote by $\delta(a)$ the derivative of a.

If k is a field and $P \in k[X]$, P' denotes the derivative of P and $Z(P)$ the set of roots of P.

Set $F := \{f : (a, \infty) \to \mathbb{C} \mid a \in \mathbb{R}\}$ and
$G := \{f : (a, \infty) \to \mathbb{R} \mid a \in \mathbb{R}\} \subseteq F$.

For $f, g \in F$ define $f \sim g$ by $\exists a \in \mathbb{R}, \forall x > a, f(x) = g(x)$. $\sim$ is an equivalence relation on F; for any $f \in F$, the class of f will be denoted by $\overline{f}$.

Set $\mathcal{F} := F/\sim$ and $\mathcal{G} := G/\sim$.

$\mathcal{F}$ and $\mathcal{G}$ are rings with operations defined by: $\overline{f+g} = \overline{f} + \overline{g}$, $\overline{f.g} = \overline{f}.\overline{g}$.

We say that $\overline{f}$ is differentiable if there exists $a \in \mathbb{R}$ such that f is differentiable on $(a, +\infty)$, and in that case we define the derivative of $\overline{f}$ as $\delta(\overline{f}) := \overline{\delta(f)}$

Definition 1
A Hardy field is a subring K of $\mathcal{G}$ which is a field and such that for every $\overline{f} \in K$, $\overline{f}$ is differentiable and $\delta(\overline{f}) \in K$.

A complex Hardy field is a subring of K of $\mathcal{F}$ which is a field and such that for every $\overline{f} \in K$, $\overline{f}$ is differentiable and $\delta(\overline{f}) \in K$.

The goal of this lecture is to describe the real closure of a Hardy field and prove that it is again a Hardy field.

Let K be a Hardy field and $P \in K[X]$ of degree n, $P = \sum_{m=0}^{n} \overline{f}_m X^m$. If $a \in \mathbb{R}$ is such that $f_1, \ldots, f_n$ are all defined and C^1 on $(a, +\infty)$ and $f_n(x) \neq 0$ for all $x > a$, we say that P is *defined* on $(a, +\infty)$. Note that such an a always exists.

If P is defined on $(a, +\infty)$, then for any $x > a$ we define P_x as the polynomial of $\mathbb{R}[X]$: $P_x = \sum_{m=0}^{n} f_m(x) X^m$; note that P_x also has degree n and that $(P_x)' = (P')_x$, which we will just denote by P'_x. Of course, the definition of P_x depends on the choice of representatives for $\overline{f}_1, \ldots, \overline{f}_n$. However, whenever a polynomial is introduced, we will always assume we have fixed the representatives of its coefficients, so that P_x is well-defined.

Note that if $g \in F$, $P(\overline{g})$ is the germ of the function $\sum f_i g^i$, so $P(\overline{g}) = 0$ if and only if there exists a such that for all $x > a$, $P_x(g(x)) = 0$.

I recall the following well-known fact:

Proposition
Let K be a field and $P \in K[X]$.

P has only simple roots in its splitting field if and only if $gcd(P, P') = 1$ if and only if there exist $A, B \in K[X]$ such that $AP + BP' = 1$.

If $char(K) = 0$ and P is irreducible then $gcd(P, P') = 1$.

The keystone of the proof of the main theorem of this lecture is a well-known theorem from analysis, namely the implicit function theorem, which I recall here:

Theorem (IFT)
Let $\mathcal{U} \subseteq \mathbb{R}^n, \mathcal{V} \subseteq \mathbb{R}^m$ two open sets, $u : \mathcal{U} \times \mathcal{V} \to \mathbb{R}^m$ a C^k function for some $k \in \mathbb{N}$ and $(x_0, y_0) \in \mathcal{U} \times \mathcal{V}$ such that $u(x_0, y_0) = 0$ and $det(\frac{\partial u}{\partial y}(x_0, y_0)) \neq 0$. Then there exists an open ball $\mathcal{U}_0$ containing x_0, an open ball $\mathcal{V}_0$ containing y_0 and a C^k function $\phi : \mathcal{U}_0 \to \mathcal{V}_0$ such that for any $(x, y) \in \mathcal{U}_0 \times \mathcal{V}_0$,

$u(x, y) = 0 \Leftrightarrow y = \phi(x)$

We will actually need a particular form of the implicit function theorem, namely:

Proposition 2 (IFT')
Let K be a Hardy field, $P \in K[X]$ defined on (a, ∞), $x_0 > a$ and y_0 a complex root of P_{x_0} which is not a root of P'_{x_0}. Then there exists an open interval I containing x_0, an open ball $\mathcal{U}$ containing y_0 and a C^1 function $\phi : I \to \mathcal{U}$ such that:

$(*) \quad \forall (x, y) \in I \times \mathcal{U}, P_x(y) = 0 \Leftrightarrow y = \phi(x)$

Proof. Set $\begin{array}{cccc} u & : & (a,\infty) \times \mathbb{C} & \to & \mathbb{C} \\ & & (x,y) & \mapsto & P_x(y) \end{array}$

u is C^1 on $(a, +\infty) \times \mathbb{C}$. By assumption, we have $u(x_0, y_0) = 0$ and $\frac{\partial u}{\partial y}(x_0, y_0) = P'_{x_0}(y_0) \neq 0$, so we can apply the IFT to the function u at the point (x_0, y_0). □

Lemma 3
Let K be a Hardy field and $P \in K[X]$ defined on $(a, +\infty)$. If $gcd(P, P') = 1$ then there exists $b \in \mathbb{R}$ such that for all $x > b$ $gcd(P_x, P'_x) = 1$.

Proof. Since $gcd(P, P') = 1$ there is $A, B \in K[X]$ such that $AP + BP' = 1$. Now let $b > a$ such that A, B are defined on (b, ∞); for $x > b$ we have $A_x P_x + B_x P'_x = 1$, hence $gcd(P_x, P'_x) = 1$. □

Lemma 4
Let K be a Hardy field, $P \in K[X]$ non-zero defined on (a, ∞) and f a continuous function from (a, ∞) to $\mathbb{C}$ such that $\forall x > a, P_x(f(x)) = 0$ and $P'_x(f(x)) \neq 0$. Then f is differentiable on (a, ∞).

Proof. Let $x_0 > a$, $y_0 := f(x_0)$. By hypothesis, y_0 is a root of P_{x_0} but not of P'_{x_0} so we can apply IFT', and we get $I, \mathcal{U}$ and ϕ as in IFT' such that $(*)$ holds.

Set $J := I \cap f^{-1}(\mathcal{U})$. $\mathcal{U}$ is a neighborhood of y_0 and f is continuous, so $f^{-1}(\mathcal{U})$ is a neighborhood of x_0, so J is also a neighborhood of x_0.

Let $x \in J$; by assumption we have $P_x(f(x)) = 0$ and $(x, f(x)) \in I \times \mathcal{U}$, which by $(*)$ implies $f(x) = \phi(x)$.

This proves that $f_{|J} = \phi_{|J}$, which since ϕ is C^1 implies that f is differentiable at x_0.

Since x_0 was chosen arbitrarily, this proves that f is differentiable on (a, ∞). □

Proposition 5
Let K be a Hardy field and $f \in F$ a continuous function such that there exists $P \in K[X]$ non-zero such that $P(\overline{f}) = 0$. Then the ring $K[\overline{f}]$ is a complex Hardy field. If f happens to be in G, then $K[\overline{f}]$ is a Hardy field.

Proof. Without loss of generality, we can assume that P is irreducible. This implies that $K[\overline{f}]$ is isomorphic to $K[X]/(PK[X])$, so it is a field. We now have to show that every element of $K[\overline{f}]$ is differentiable and that $K[\overline{f}]$ is stable under derivation. It is sufficient to show that $\overline{f}$ is differentiable and that $\delta(\overline{f}) \in K[\overline{f}]$.

Since $P(\overline{f}) = 0$, there exists $a \in \mathbb{R}$ such that for all $x > a$, $P_x(f(x)) = 0$. Since P is irreducible and $char(K) = 0$, $gcd(P, P') = 1$ so by lemma 3 there exists $a > b$ such that for all $x > b \quad gcd(P_x, P'_x) = 1$, so that P_x and P'_x have no root in common. Thus, for any $x > b$, $P_x(f(x)) = 0 \neq P'_x(f(x))$. We can apply lemma 4 and obtain that f is differentiable on $(b, +\infty)$.

Set $P = \sum_{m=0}^{n} \overline{g}_m X^m$.

$$
\begin{aligned}
0 = \delta(P(\overline{f})) &= \sum_{m=0}^{n} \delta(\overline{g}_m \overline{f}^m) \\
&= \delta(\overline{g_0}) + \sum_{m=1}^{n} (\delta(\overline{g}_m)\overline{f}^m + m\overline{g}_m \overline{f}^{m-1}\delta(\overline{f})) \\
&= \sum_{m=0}^{n} \overline{\delta(g_m)}\overline{f}^m + \delta(\overline{f}) \sum_{m=1}^{n} m\overline{g}_m \overline{f}^{m-1} \\
&= Q(\overline{f}) + \delta(\overline{f})P'(\overline{f})
\end{aligned}
$$

with $Q \in K[X]$, hence $\delta(\overline{f}) = \frac{-Q(\overline{f})}{P'(\overline{f})} \in K[\overline{f}]$. $\square$

Lemma 6
Let K be a Hardy field, $n \in \mathbb{N}$ and $P \in K[X]$ of degree n defined on (a, ∞) such that for all $x > a$, P_x has n distinct roots in $\mathbb{C}$.

For any pair $(x_0, y_0) \in (a, +\infty) \times \mathbb{C}$ such that y_0 is a root of P_{x_0}, there exists a C^1 function $\phi : (a, \infty) \to \mathbb{C}$ such that $y_0 = \phi(x_0)$ and

$$\forall x > a \quad P_x(\phi(x)) = 0 \quad (\dagger)$$

Proof. Let $x_0 > a$ and y_0 a complex root of P_{x_0}. Since P_{x_0} has simple roots, y_0 is not a root of P'_{x_0} so we can apply IFT' and we get an open interval I containing x_0, an open ball $\mathcal{U}$ containing y_0 and a C^1 function $\phi : I \to \mathcal{U}$ such that $(*)$ is satisfied, which in particular implies that $\phi(x_0) = y_0$ and $P_x(\phi(x)) = 0$ for all $x \in I$.

Let $\mathcal{E} := \{(J, \psi) \mid J$ open interval $, I \subseteq J, \psi$ is a C^1-extension of ϕ to J and satisfies $(\dagger)$ on $J\}$
$\mathcal{E}$ is non-empty since $(I, \phi) \in \mathcal{E}$.

We can order $\mathcal{E}$ by saying that $(J, \psi) \leq (J', \chi)$ if $J \subseteq J'$ and χ extends ψ.

Let $(J_h, \psi_h)_{h \in H}$ be a chain in $\mathcal{E}$. Set $J := \bigcup_{h \in H} J_h$ and define ψ on J by $\psi(x) = \psi_h(x)$ if $x \in J_h$; this definition makes sense because ψ_h is an extension of $\psi_{h'}$ for any $h, h' \in H$ such that $J_{h'} \subseteq J_h$. If $x \in J$, then $x \in J_h$ for some $h \in H$, and since $(J_h, \psi_h) \in \mathcal{E}$ we have $P_x(\psi_h(x)) = 0$ hence $P_x(\psi(x)) = 0$. Thus, ψ satisfies $(\dagger)$ on J, so $(J, \psi) \in \mathcal{E}$. Moreover, we have $(J_h, \psi_h) \leq (J, \psi)$ for any $h \in H$, so (J, ψ) is an upper bound of $(J_h, \psi_h)_{h \in H}$.

We just proved that any chain of $\mathcal{E}$ has an upper bound. By Zorn's lemma, it follows that $\mathcal{E}$ has a maximal element (J, ψ)

To conclude the proof, we only have to show that $J = (a, +\infty)$

Set $b := supJ$. Towards a contradiciton, assume that $b \neq +\infty$. By hypothesis, P_b has n distinct roots $y_1, \ldots, y_n$, none of which is a root of P'_b. We can apply IFT' again at each of the points $(b, y_1), \dots, (b, y_n)$, and we get open intervals $I_1, \dots, I_n$ containing b, open balls $\mathcal{U}_1, \dots, \mathcal{U}_n$ containing $y_1, \dots, y_n$ and $\phi_1 : I_1 \to \mathcal{U}_1, \dots, \phi_n : I_n \to \mathcal{U}_n$ such that for each $m \in \{1, \dots, n\}$, for any $(x, y) \in I_m \times \mathcal{U}_m$, $P_x(y) = 0 \Leftrightarrow y = \phi_m(x)$

Since $y_1, \dots, y_n$ are pariwise distinct,we can choose the sets $\mathcal{U}_1, \dots, \mathcal{U}_n$ so small that they are pairwise disjoint.

Now let $I' := \bigcap_{m=1}^n I_m$. For any $x \in I'$, we have $\phi_1(x) \in \mathcal{U}_1, \dots, \phi_n(x) \in \mathcal{U}_n$; since $\mathcal{U}_1, \dots, \mathcal{U}_n$ are pairwise disjoint, $\phi_1(x), \dots, \phi_n(x)$ are pairwise distinct. By $(*)$, each $\phi_m(x)$ is a root of P_x; since P_x has n roots, it follows that $Z(P_x) = \{\phi_1(x), \dots, \phi_n(x)\} \subseteq \bigcup_{m=1}^n \mathcal{U}_m$.

Now let $J' := I' \cap J$; note that J' is an interval. For any $x \in J'$, $(\dagger)$ implies that $\psi(x)$ is a root of P_x, hence $\psi(x) \in \bigcup_{m=1}^n \mathcal{U}_m$. Thus, we have $\psi(J') \subseteq \bigcup_{m=1}^n \mathcal{U}_m$. Since ψ is continuous, $\psi(J')$ is connected. Since $\mathcal{U}_1, \dots, \mathcal{U}_n$ are pariwise disjoint, this implies that there exists $m \in \{1, \dots, n\}$ such that $\psi(J') \subset \mathcal{U}_m$.

Let $x \in J'$; we have $(x, \psi(x)) \in I_m \times \mathcal{U}_m$ and $P_x(\psi(x)) = 0$. Since ϕ_m satisfies $(*)$ on $I_m \times \mathcal{U}_m$, it follows that $\psi(x) = \phi_m(x)$. This proves that $\psi_{|J'} = \phi_{m|J'}$.

Define the function $\tilde{\psi}$ on $J \cup I'$ by: $\tilde{\psi}(x) = \begin{cases} \psi(x) & \text{if } x \in J \\ \phi_m(x) & \text{if } x \in I' \end{cases}$

This definition makes sense because ψ and ϕ_m agree on I'. $\tilde{\psi}$ is a strict extension of ψ. Since ψ and ϕ_m are C^1, $\tilde{\psi}$ is also C^1. Since ψ satisfies $(\dagger)$ on J and ϕ_m satisfies $(*)$ on I', it follows that $\tilde{\psi}$ satisfies $(\dagger)$ on $J \cup I'$, which contradicts the maximality of (J, ψ).

Thus, $b = +\infty$.

We could prove the same way that $inf J = a$.

□

Lemma 7
Let K be a Hardy field and $P \in K[X]$ of degree n such that $gcd(P, P') = 1$.

There exist $a \in \mathbb{R}$ and n C^1 functions $\phi_1, \dots \phi_n : (a, \infty) \to \mathbb{C}$ such that for each $x > a$, $Z(P_x) = \{\phi_1(x), \dots, \phi_n(x)\}$.

Proof. By lemma 3, there exists $a_0 \in \mathbb{R}$ such that for all $x > a_0$ $gcd(P_x, P'_x) = 1$ which means that P_x has n distinct roots in $\mathbb{C}$.

Let $a > a_0$, and let $y_1, \dots, y_n$ be the n distinct roots of P_a. By the previous lemma, we get n C^1 functions $\phi_1, \dots, \phi_n$ from (a_0, ∞) to $\mathbb{C}$ such that for any $m \in \{1, \dots, n\}, \phi_m(a) = y_m$ and for any $x > a$, $\{\phi_1(x) \dots, \phi_n(x)\} \subseteq Z(P_x)$. To show equality, we just have to show that $\phi_l(x) \neq \phi_m(x)$ for any $x > a$ and any $m, l \in \{1, \dots, n\}$.

Now let $m, l \in \{1 \dots n\}$ and $E := [a, \infty] \cap (\phi_m - \phi_l)^{-1}(\{0\})$. Assume $E \neq \varnothing$.

By continuity of ϕ_m and ϕ_l, E is a closed subset of $\mathbb{R}$ and has a lower bound a, so it has a minimum b. Since $\phi_m(a) \neq \phi_l(a)$, $b > a$.

Set $c := \phi_m(b)$. c is a root of P_b, so we can apply IFT' at the point (b, c) and we get an open neighborhood $I \times \mathcal{U}$ of (b, c) and a map $\phi : I \to \mathcal{U}$ satisfying $(*)$. Since $\mathcal{U}$ is a neighborhood of c, and since $c = \phi_m(b) = \phi_l(b)$, $\phi_l^{-1}(\mathcal{U})$ and $\phi_m^{-1}(\mathcal{U})$ are neighborhoods of b, so $J := I \cap (a, \infty) \cap \phi_l^{-1}(\mathcal{U}) \cap \phi_m^{-1}(\mathcal{U})$ is a neighborhood of b. Let $x \in J$ such that $x < b$; $(x, \phi_l(x))$ and $(x, \phi_m(x))$ both belong to $I \times \mathcal{U}$ and we have $P_x(\phi_m(x)) = P_x(\phi_l(x)) = 0$; since ϕ satisfies $(*)$ on $I \times \mathcal{U}$, this implies $\phi_l(x) = \phi(x) = \phi_m(x)$, so $x \in E$, which contradicts the minimality of b.

Thus, $E = \varnothing$. □

Proposition 8
Let k be a Hardy field,

$K := \{\overline{f} \in \mathcal{G} \mid$ **f is continuous and** $\exists P \in k[X]$ **such that** $P \neq 0 \wedge P(\overline{f}) = 0\}$ **and** $L := \{\overline{f} \in \mathcal{F} \mid$ **f is continuous and** $\exists P \in k[X]$ **such that** $P \neq 0 \wedge P(\overline{f}) = 0\}$**.**

Then K is a Hardy field, L is a complex Hardy field, L is the algebraic closure of k and K is the real closure of k.

Proof. Obviously, $k \subseteq K \subseteq L$.

Let $\overline{f}, \overline{g} \in K$. By proposition 5, $k[\overline{f}]$ is a Hardy field. Since g is continuous and $\overline{g}$ is canceled by a polynomial in $k[\overline{f}][X]$, we can again use proposition 5 and we get that $k[\overline{f}, \overline{g}]$ is a Hardy field, and since this field is algebraic over k it is contained in K. Since $k[\overline{f}, \overline{g}]$ is a Hardy field, we have $0, 1, \overline{f} - \overline{g}, \frac{\overline{f}}{\overline{g}}, \delta(\overline{f}), \delta(\overline{g}) \in k[\overline{f}, \overline{g}]$, hence $0, 1, \overline{f} - \overline{g}, \frac{\overline{f}}{\overline{g}}, \delta(\overline{f}), \delta(\overline{g}) \in K$. This proves that K is Hardy field. The same proof shows that L is a complex Hardy field.

Now let us show that L is algebraically closed. Let $P \in k[x]$ irreducible of degree $n > 1$. Since $char(k) = 0$, $gcd(P, P') = 1$. By 7 there is $a \in \mathbb{R}$ and C^1 functions $\phi_1, \dots, \phi_n : (a, +\infty) \to \mathbb{C}$ such that for any $x > a$, $Z(P_x) = \{\phi_1(x), \dots, \phi_n(x)\}$. This means that $\overline{\phi}_1, \dots, \overline{\phi}_n$ are n distinct roots of P. Since $\phi_1, \dots, \phi_n$ are continuous functions from $(a, +\infty)$ to $\mathbb{C}$ and $\overline{\phi}_1, \dots, \overline{\phi}_n$ are canceled by $P \in k[X]$, we have $\overline{\phi}_1, \dots \overline{\phi}_n \in L$

Thus, any polynomial with coefficients in k splits in L. Since L/k is an algebraic extension, this proves that L is algebraically closed, and thus L is the algebraic closure of k.

Now note that $L = K(i)$. Since $K(i)$ is algebraically closed, K is real closed, and it is the real closure of k.

□

Corollary 9
The real closure of a Hardy field is a Hardy field.