Ordinal numbers

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Introduction

Consider the class WO of all well-ordered sets; if we denote by $\cong$ the relation "being isomorphic to" between ordered structres, then $\cong$ defines an equivalence relation on WO. An ordinal can be thought of as an equivalence class of WO under the relation $\cong$; more precisely, the class *Ord* of all ordinals satisfy the property that, for any well-ordered set A, there exists exactly one ordinal isomorphic to A.

Another way to consider ordinals is to see them as an ordered sequence continuing the sequence of natural numbers. Remember that in set theory, we define natural numbers as follows:

$$
\begin{aligned}
&0 := \varnothing \\
&1 := \{0\} \\
&2 := \{0, 1\} \\
&3 := \{0, 1, 2\} \\
&\vdots \\
&n + 1 := \{0, 1, \ldots, n\} = n \cup \{n\} \\
&\vdots \\
&\omega := \bigcup_{n \text{ natural number}} n
\end{aligned}
$$

We can continue this process by defining the successor $\omega + 1 := \omega \cup \{\omega\}$ of ω, and then $\omega + 2$ the successor of $\omega + 1$, and so on, and after repeating this ω times we can define $\omega + \omega := \bigcup_{n\in\omega}(\omega + n)$. Repeating this process indefinitely, we build the whole class of ordinals, which consists of infinitely many successive copies of ω; see picture on page 6 for a better understanding.

1 Preliminaries

Notation: If A and B are ordered sets, $A \hookrightarrow B$ means that A is embeddable into B, i.e there exists an order-preserving injective map from A to B.

Ordinals are a particular kind of well-ordered structures, which is why I need to recall a few facts about well-orderings.

First, recall that the induction principle which is well-known for integers can be generalized to well-ordered sets:

Theorem 1.1 (Transfinite Induction)
Let $(A, <)$ be a well-ordered set and $\mathcal{P}(x)$ a property defined on A satisfying:

$\forall a \in A, ((\forall b < a \mathcal{P}(b)) \Rightarrow \mathcal{P}(a))$

Then $\mathcal{P}(a)$ is true for every $a \in A$.

Proof. Consider $B := \{a \in A \mid \mathcal{P}(a) \text{ is not true }\}$.

Assume $B \neq \varnothing$; since A is well-ordered, we can consider $a := min(B)$. Then $\mathcal{P}(b)$ is true for every $b < a$, but $\mathcal{P}(a)$ is false, which contradicts the hypothesis of the theorem.

Thus, $B = \varnothing$ □

Definition 1.2
An initial segment of A is a subset of A of the form $A_a := \{b \in A \mid b \leq a\}$.

Proposition 1.3
Let $(A, <)$ be a well-ordered set. If B is a proper initial segment of A, then there is no embedding $A \hookrightarrow B$. In particular, A and B are not isomorphic.

Proof. Assume there exists an embedding $f : A \hookrightarrow B$.

We prove by induction on A: for all $x \in A$, $f(x) \geq x$.

Let $a \in A$ and assume that for all $b < a$, $f(b) \geq b$.

Let $b \in A$ such that $b < a$. Since f preserves the order, we have $f(b) < f(a)$, and by induction hypothesis we also have $f(b) \geq b$ hence $f(a) > b$.

This proves that for all $b < a$, $f(a) > b$, hence $f(a) \geq a$.

Since B is a proper subset of A, there exists $a \in A \backslash B$, and since B is an initial segment of A we then have $a > b$ for all $b \in B$; in particular $a > f(a)$, hence a contradiction.

□

We now introduce the notion of transitive set which plays a central role in the definition of ordinals

Definition 1.4
A set A is called transitive if every element of A is also a subset of A.

Equivalently: A is transitive if and only if: for all $x \in A$, for all $y \in x$, $y \in A$.

Lemma 1.5
Let A be a transitive set. Then $\in$ is a transitive relation on A if and only if for every $a \in A$, a is a transitive set.

Proof. Assume $\in$ is transitive and let $a \in A$. We want to prove that a is a transitive set. Let $x \in y \in a$; since A is a transitive set, we have $x \in A$, and so $y \in A$ too. Since $\in$ is a transitive relation on A, the relation $x \in y \in a$ implies $x \in a$. This proves that a is a transitive set.

Conversely, assume that a is a transitive set for all $a \in A$.

Let $a, b, c \in A$ such that $a \in b \in c$. Since c is a transitive set, this relation implies $a \in c$. □

Lemma 1.6
A union of transitive sets is a transitive set.

Proof. Let $(A_i)_{i \in I}$ be a family of transitive sets and set $A := \bigcup_{i \in I} A_i$. We want to show that A is transitive.

Let $a \in A$ and $x \in a$. There exists $i \in I$ suchtthat $a \in A_i$. Since A_i is a transitive set, the relation $x \in a \in A_i$ implies $x \in A_i$, hence $x \in A$. □

2 Ordinals: definition and basic properties

Definition 2.1
A set α is called an ordinal if

- **α is a transitive set**
- **$(\alpha, \in)$ is a well-ordered set**

Remark 2.2
- The class *Ord* of all ordinals is not a set in the sense of axiomatic set theory.
- The definition above implies in particular that $\in$ is an order on α, so it is a transitive relation. According to lemma 1.5, this means that any element of α is a transitive set.

Example 2.3
Every natural number is an ordinal, and so is ω.

Proposition 2.4
$\in$ defines a strict order on *Ord*.

Proof.
- $\in$ is transitive: let $\alpha \in \beta \in \gamma$ all in *Ord*. Since γ is a transitive set, we have $\alpha \in \gamma$.
- $\in$ is antisymmetric: Assume there exists $\alpha, \beta \in Ord$ such that $\beta \in \alpha \in \beta$. Since β is a transitive set, we have $\beta \in \beta$, and since $\beta \in \alpha \in \beta$, the relation $\in$ is not antisymmetric on β: this is a contradiction to the fact that β is an ordinal.

□

The order we consider on *Ord* will always be the one given by $\in$; thus, if α, β are ordinals, $\alpha < \beta$ means $\alpha \in \beta$. I will use both notations indifferently.

Proposition 2.5
Let α be an ordinal. Then $\alpha := \{\beta \mid \beta$ is an ordinal and $\beta < \alpha\}$.

Proof. Let $\beta \in \alpha$, we want to show that β is an ordinal.

By remark 2.2, we know that β is a transitive set.

Since α is a transitive set, we have $\beta \subseteq \alpha$, so the relation $\in$ defined on β is the restriction of the relation $\in$ defined on α. Since $(\alpha, \in)$ is well-ordered, this implies that $(\beta, \in)$ is well-ordered.

Thus, β is an ordinal. □

As immediate corollaries we have:

Corollary 2.6
Let $\alpha, \beta \in Ord$.

$\alpha \subseteq \beta$ **if and only if** $\forall \delta \in Ord, \delta < \alpha \Rightarrow \delta < \beta$.

$\alpha = \beta$ **if and only if** $\forall \delta \in Ord, \delta < \alpha \Leftrightarrow \delta < \beta$.

Corollary 2.7
Let $\alpha, \beta \in Ord$ such that $\alpha < \beta$. Then α is a proper initial segment of β.

Our next step is to show that the order on ordinals is total.

Lemma 2.8
Let α, β be ordinals such that $\beta \nsubseteq \alpha$. Then $\gamma := min(\beta \backslash \alpha)$ exists and is included in α.

If moreover $\alpha \subset \beta$, then $\gamma = \alpha$, and so $\alpha \in \beta$.

Proof. The existence of γ comes from the fact that $\beta\backslash\alpha \neq \varnothing$ and that β is well-ordered. Note that since $\gamma \in \beta$, γ is an ordinal and $\gamma < \beta$.

Let $\delta < \gamma$. Since $\gamma < \beta$, we have $\delta \in \beta$; however, since $\delta < \gamma$, we have by minimality of γ: $\delta \notin \beta\backslash\alpha$, hence $\delta \in \alpha$. This proves that $\gamma \subseteq \alpha$.

Now assume that $\alpha \subset \beta$ and let $\delta < \alpha$; we also have $\delta \in \beta$. If $\delta > \gamma$, we would have $\alpha > \gamma$, i.e $\gamma \in \alpha$, which by definition of γ is impossible. Since $\delta, \gamma \in \beta$, and β is totally ordered, this implies $\delta < \gamma$. This proves that $\alpha \subseteq \gamma$,hence $\gamma = \alpha$. □

Lemma 2.9
Let α, β be ordinals. Then $\alpha \leq \beta \Leftrightarrow \alpha \subseteq \beta$.

Proof. $\Rightarrow$: if $\alpha = \beta$ there is nothing to prove; if $\alpha < \beta$, the fact that β is a transitive set implies $\alpha \subseteq \beta$.

$\Leftarrow$: Assume $\alpha \subsetneq \beta$. In that case lemma 2.8 implies that $\alpha \in \beta$,i.e $\alpha < \beta$.

□

Proposition 2.10
$<$ (which is also $\in$) is a total order on *Ord*

Proof. Let α, β be ordinals such that $\beta \nleq \alpha$. By lemma 2.9, we have $\beta \nsubseteq \alpha$, which by lemma 2.8 implies $\gamma := min(\beta\backslash\alpha) \subseteq \alpha$. By lemma 2.9, we have $\gamma \leq \alpha$; however, by definition of γ, we can't have $\gamma \in \alpha$, hence $\gamma = \alpha$, hence $\alpha \in \beta$. □

Proposition 2.11
If $\alpha \neq \beta$, then α and β are not isomorphic.

Proof. Since $<$ is a total order, we can assume $\alpha < \beta$. Then α is a proper initial segement of β, which by proposition 1.3 implies that α and β are not isomorphic. □

Proposition 2.12
$(Ord, <)$ is well-ordered.

Proof. Since the order is total, we just have to show that there is no strictly decreasing infinite sequence of ordinals $\alpha_0 > \alpha_1 > \alpha_2 > \cdots > \alpha_n > \ldots$. But if such a sequence existed, then $\alpha_n \in \alpha_0$ for every $n > 0$, so $(\alpha_n)_{n>0}$ would be an infinite decreasing sequence of elements of α_0, which would contradict the fact that α_0 is well-ordered. □

Proposition 2.13

- **If α is an ordinal, then so is $\alpha \cup \{\alpha\}$.**

  $\alpha + 1 := \alpha \cup \{\alpha\}$ is called the successor of α.

- **If A is a set of ordinals, then $\bigcup A$ is an ordinal.**

  $sup(A) := \bigcup A$ is the supremum of A (i.e, it is the smallest ordinal bigger than every element of A.)

Proof. Set $\delta := \bigcup A$. δ is a union of transitive sets so by lemma 1.6 it is a transitive set. To show that δ is well-ordered, just note that $\delta \subset Ord$, and that Ord is well-ordered.

Let us show that δ is the supremum of A:

clearly, $\delta > \alpha$ for any $\alpha \in A$. Let $\gamma \in Ord$ such that $\gamma > \alpha$ for all $\alpha \in A$. Let $\beta \in \delta$; there exists $\alpha \in A$ such that $\beta \in \alpha < \gamma$, hence $\beta \in \gamma$. This proves that $\gamma \subseteq \delta$, hence $\gamma \leq \delta$. □

Remark 2.14

- The definition of the successor of an ordinal is consistent with the usual definition of the successor of an integer: indeed, if $n \in \omega$, then $n+1 = \{0, 1, \ldots, n\} = n \cup \{n\}$.
- $\alpha + 1$ is the smallest ordinal strictly bigger than α.
- $sup(A)$ is not necessarily a max: take $A := \{2n \mid n \in \omega\}$, then $sup(A) = \omega$, but A has no max.
- However, if we take $A := \{0, 1, 3\}$, then $sup(A) = max(A) = 3$.
- If α is an ordinal, then in particular it is a set of ordinals, and in that case we have $sup\alpha = \alpha$.

Definition 2.15
An ordinal which is not a successor and is not 0 is called a limit ordinal.

Example 2.16
ω is a limit ordinal (it is actually the smallest one).

Thus, we can say that there are three kinds of ordinals: 0, successor ordinals and limit ordinals. The distinction bewteen limit and successor ordinals is an important one, since they have different properties; for example, a successor ordinal has a max, but a limit ordinal does not. We will also see that we usually seperate the case of successor and limit ordinal when making a proof by induction on ordinals.

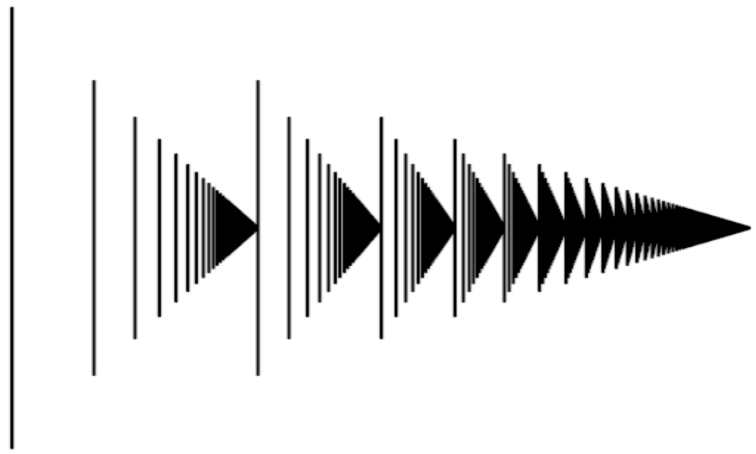
Proposition 2.13 gives us the tools to inductively construct ordinals. Remember that natural numbers are constructed by starting with 0 and by then repeatedly applying the successor map: we define 1 as the successor of 0, 2 as the the successor of 1, and so on.

Ordinals are constructed by alternately applying these two operations:

- Taking the successor of the last ordinal defined.
- once the successor operation has been repeated ω times, take the supremum of all the already defined ordinals.

More prescisely: we start by defining 0, then apply the successor operation ω times to construct the set of natural numbers. We then define ω as the supremum of all natural numbers. We then repeat the same process: after ω comes its successor $\omega+1 := \omega \cup \{\omega\}$, then $\omega+2 := (\omega+1) \cup \{\omega+1\}$, and so on; after applying the successor operation ω times, we arrive at $\omega + \omega := sup_{n \in \omega}(\omega + n)$. By repeating this process indefinitely, we construct the class of ordinals.

To help you visualize this, here is a matchstick representation of the ordinal ω^2; each stick represents an ordinal:



We now give another version of theorem 1.1, used for ordinals. Since there are three kinds of ordinals (0, successor ordinal, and limit ordinal), the induction is split into three cases:

Theorem 2.17 (Transfinite induction on *Ord*)
Let $\mathcal{P}(x)$ be a property defined on ordinals such that :

- **$\mathcal{P}(0)$ is true.**
- **If $\mathcal{P}(\alpha)$ is true, then $\mathcal{P}(\alpha + 1)$ is true.**
- **If α is a limit ordinal and if $\mathcal{P}(\beta)$ is true for every $\beta < \alpha$ then $\mathcal{P}(\alpha)$ is true.**

Then $\mathcal{P}(\alpha)$ is true for every $\alpha \in Ord$.

Theorem 2.18 (Transfinite induction on an ordinal)
Let $\alpha \in Ord$ and $\mathcal{P}(x)$ a property defined on α such that :

- **$\mathcal{P}(0)$ is true.**
- **If $\beta + 1 < \alpha$ and $\mathcal{P}(\beta)$ is true, then $\mathcal{P}(\beta + 1)$ is true.**
- **If $\beta \in \alpha$ is a limit ordinal and if $\mathcal{P}(\gamma)$ is true for every $\gamma < \beta$ then $\mathcal{P}(\beta)$ is true.**

Then $\mathcal{P}(\beta)$ is true for every $\beta \in \alpha$.

We now come to the main theorem:

Theorem 2.19
Let $(A, <)$ be a well-ordered set.
There exists a unique ordinal α and a unique isomorphism $\pi : A \to \alpha$.
α is called the order type of $(A, <)$, denoted $ot(A)$

The proof of this theorem will make use of the two following lemmas:

Lemma 2.20
Let $(A, <)$ be well-ordered. If there exists $\beta \in Ord$ such that $A \hookrightarrow \beta$, then there exists a unique isomorphism $\pi : A \to \alpha := min\{\beta \in Ord \mid A \hookrightarrow \beta\}$.

Proof. We build the isomorphism by induction on α:

- $\pi(0) := minA$
- Assume π has been constructed up to β, so π is an isomorphism from $\beta + 1$ to $A_{\pi(\beta)}$. If $\beta + 1 = \alpha$, we are done.

  Assume $\beta+1 < \alpha$. If $A = A_{\pi(\beta)}$, we would have an embedding $\pi^{-1} : A \hookrightarrow \beta+1$, which would contradict the minimality of α, so $A \neq A_{\pi(\beta)}$. Thus, we can set: $\pi(\beta+1) := min(A \backslash A_{\pi(\beta)})$.
- Let β be a limit ordinal such that for all $\gamma < \beta$, $\pi(\gamma)$ is already defined, so that we have an isomorphism $\pi : \beta \to B$, where $B := \cup\{\pi(\gamma) \mid \gamma < \beta\}$. If $\beta = \alpha$, we are done.

  Assume $\beta < \alpha$; If $B = A$, A would be isomorphic to β, which would contradict the minimality of α. Thus, B is a proper subset of A and we can define $\pi(\beta) := min(A \backslash B)$.

By construction, it is easy to see that π in injective and preserves the order. Assume it is not surjective; then α is isomorphic to a proper initial segment of A, so A cannot be embedded into α: contradiction. Thus, π is surjective.

You can show the uniqueness of π like this: consider another isomorphism $\phi : \alpha \to A$ and show by induction on α that $\pi = \phi$.

□

Lemma 2.21
Let $(A, <)$ be well-ordered. Assume that for all $a \in A$, there exists $\beta_a \in Ord$ such that $A_a \hookrightarrow \beta_a$. Then there exists $\alpha \in Ord$ such that $A \hookrightarrow \alpha$.

Proof. For each $a \in A$ set $\alpha_a := min\{\beta \in Ord \mid A_a \hookrightarrow \beta\}$. Let us show that the map $a \to \alpha_a$ is an embedding of A into $\alpha := sup\{\alpha_a \mid a \in A\} + 1$:

Let $a, b \in A$ such that $a < b$. Assume $\alpha_b \leq \alpha_a$; in that case, α_b is an initial segment of α_a. Moreover, by lemma 2.20, A_a is isomorphic to α_a, so there is an embedding $\alpha_a \hookrightarrow A_a$. Thus, we have a sequence of embeddings: $A_b \hookrightarrow \alpha_b \hookrightarrow \alpha_a \hookrightarrow A_a$, hence $A_b \hookrightarrow A_a$. But since $a < b$, A_a is a proper initial segment of A_b, so we have a contradiction with lemma 1.3. This proves $\alpha_a < \alpha_b$. □

proof of the theorem. Note that the unicity of α is given by proposition 2.11

By lemma 2.20, it is sufficient to prove that A is embedded into an ordinal.

We are going to prove by induction on A the following: for any $a \in A$, there is an embedding $A_a \hookrightarrow \alpha_a \in Ord$, and we will conclude by lemma 2.21

Let $a \in A$ and assume that for all $b < a$, there is an embedding $A_b \hookrightarrow \alpha_b$. set $B := A_a \backslash \{a\}$; this is a well-ordered set which satisfies the condition of lemma 2.21, so there exists an ordinal α such that we have an embedding $\pi : B \hookrightarrow \alpha$. We can extend π to A_a by setting $\pi(a) := \alpha$, and π thus becomes an embedding from A_a into $\alpha + 1$. □

3 Arithmetic of ordinals

`Remark 3.1`
`In the exercise sheet, an alternative definition of addition and multiplication will be given; it is equivalent to the one I give here.`

Definition 3.2
Let α, β be ordinals. We define $\alpha + \beta$ by induction on β:

- $\alpha + 0 = \alpha$

- $\alpha + (\beta + 1) = (\alpha + \beta) + 1$
- **If β is a limit ordinal, then $\alpha + \beta := sup_{\gamma<\beta}(\alpha + \gamma)$**

Proposition 3.3
For any $\alpha, \beta, \gamma \in Ord$:

- $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$
- **If $\gamma < \beta$ then $\alpha + \gamma < \alpha + \beta$**

Proof. By induction on γ:

- $\alpha + (\beta + 0) = \alpha + \beta = (\alpha + \beta) + 0$
- $$\begin{aligned} & \alpha + (\beta + (\gamma + 1)) \\ &= \alpha + ((\beta + \gamma) + 1) \\ &= (\alpha + (\beta + \gamma)) + 1 \\ &= ((\alpha + \beta) + \gamma) + 1 \text{(by induction hypothesis)} \\ &= (\alpha + \beta) + (\gamma + 1) \end{aligned}$$
- If γ is a limit ordinal:
$$\begin{aligned} & \alpha + (\beta + \gamma) \\ &= \alpha + sup_{\delta<\gamma}(\beta + \delta) \\ &= sup_{\delta<\gamma}(\alpha + (\beta + \delta)) \\ &= sup_{\delta<\gamma}((\alpha + \beta) + \delta) \text{(by induction hypothesis)} \\ &= (\alpha + \beta) + \gamma \end{aligned}$$

The second claim can also be proved by induction. □

Definition 3.4
We define $\alpha.\beta$ by induction on β:

- $\alpha.0 = 0$
- $\alpha.(\beta + 1) = \alpha.\beta + \alpha$
- **If β is a limit ordinal, $\alpha.\beta := sup_{\gamma<\beta}(\alpha.\gamma)$.**

Definition 3.5
We define α^β by induction on β:

- $\alpha^0 = 1$
- $\alpha^{\beta+1} = \alpha^\beta.\alpha$
- **if β is a limit ordinal, then $\alpha^\beta = sup_{\gamma<\beta}(\alpha^\gamma)$.**

Proposition 3.6
For any $\alpha, \beta, \gamma \in Ord$:

- $\alpha.(\beta + \gamma) = \alpha.\beta + \alpha.\gamma$
- $\alpha^{\beta+\gamma} = \alpha^{\beta}.\alpha^{\gamma}$
- $(\alpha^{\beta})^{\gamma} = \alpha^{\beta.\gamma}$

Proof. All proofs are done by induction on γ. □

`Remark 3.7` • `None of these three operations are commutative.`

- $(\omega + 1).2 \neq \omega.2 + 1.2$
- $(\omega.2)^2 \neq \omega^2.2^2$

`In other words, not every rule which holds for intergers is true in general for ordinals; one should thus be careful when manipulating ordinal operations.`

Examples of computation:
$(\omega + 1).2 = (\omega + 1).(1 + 1) = (\omega + 1).1 + \omega + 1 = \omega + 1 + \omega + 1 = \omega.2 + 1$
$(\omega.2)^2 = (\omega.2)^1.\omega.2 = \omega.2.\omega.2 = \omega^2.2$
$(\omega+1).\omega = sup_{n\in\omega}((\omega+1).n)$. We can show by induction on integers that $(\omega+1).n = \omega.n+1$, hence $sup_{n\in\omega}((\omega + 1).n) = \omega^2$.