Introduction
With a differential operator $A(x,D)$ of order $m\in \mathbb{N}$, boundary operators $B_i(x,D)$ of order $m_i<2m$ for $i=1,\ldots,m$, and a domain $V \subset \mathbb{R}^n$ with compact boundary of class $C^m$, let us consider the boundary value problem
\[
\alpha u + A(x, D)u + A_i(x, D_i)u + A_{ij}(x, D_i D_j)u = f
\]
be parameter-elliptic in $V$. For $1<p<\infty$ and a UMD Banach space $F$, its $L^p(\Omega,F)$-realization $A_{\Omega}$ with $D(A)=\{u \in W^{m,p}(\Omega,F) : B_i(u)=0\}$ is known to admit an $\mathcal{R}$-bounded $\mathcal{R}^\infty$-calculus [2] if $F$ additionally enjoys property $(\mathcal{M})$.

With $m, m_i \in \mathbb{N}$, we extend the boundary value problem above to the domain $\Omega = \mathbb{R}^n \times (0,2\pi)^m$ as well as periodic boundary conditions with respect to $(0,2\pi)^m$.

By means of operator-valued Fourier multiplier results, we transfer the property of an $\mathcal{R}$-bounded $\mathcal{R}^\infty$-calculus to the $L^p(\Omega,F)$-realization of this extension.

Main theorem
Consider the boundary value problem
\[
\lambda u + A_i(x, D_i)u + A_{ij}(x, D_i D_j)u = f
\]
be parameter-elliptic in $\mathbb{R}^n \times (0,2\pi)^m \times V$. For $1<p<\infty$, and $\lambda \in \mathbb{R}$, we denote by $\mathcal{R}^\infty((0,2\pi)^m \times V)$ the set of locally bounded operators $T$ such that $T(0,2\pi)^m \times V \rightarrow (0,2\pi)^m \times V$, $T_{ij}(x,D_i,D_j)$ is the operator defined by Hill's formula
\[
\Lambda_{ij}(x,D_i,D_j)u = \left( \begin{array}{c} \frac{\partial u}{\partial x_i} \\ \frac{\partial u}{\partial x_j} \end{array} \right)
\]
for $u \in \mathbb{R}^n \times (0,2\pi)^m \times V$.

We define its $L^p(\Omega,F)$-realization $A_{\Omega}$ by
\[
D(A_{\Omega}) = \{u \in W^{m,p}(\Omega,F) : B_i(u) = 0, D_{ij}(u) = 0, \lambda u \in D(A_{\Omega}) \}
\]
for $u \in (0,2\pi)^m \times V$.

Theorem. Let $1 < p < \infty$, let $V \subset \mathbb{R}^n$ have compact boundary of class $C^m$ and let the UMD Banach space $F$ enjoy property $(\mathcal{M})$. Under suitable assumptions on the coefficients, we further assume the extended boundary value problem $A_{\Omega}(u) = (A_i + A_j + A_{ij})u$ to be parameter-elliptic in $\mathbb{R}^n \times (0,2\pi)^m \times V$.

For every $\phi \in C_c(\mathbb{R})$ with $\lambda \in \Sigma_{m-\sigma}$, there exist $\delta(\phi) > 0$ such that $A_{\Omega} + \delta$ admits an $\mathcal{R}$-bounded $\mathcal{R}^\infty$-calculus in $L^p(\Omega,F)$ with $\mathcal{R}^\infty(0,2\pi)^m \times V$.

Outline of the proof
We focus on the proof of the $\mathcal{R}$-boundedness condition (3) for $A_{\Omega}$, the $\mathcal{R}$-boundedness condition (1) can be proven along the same steps.

First consider $\alpha = 0$, i.e. $A_i(x, D_i)u$ does not exist.

a) Assume $A_{ij}(x, D_i D_j)u = 0$ to be given as principal part operator with constant coefficients. Equation (2) formally reads as
\[
\mathcal{T}_{ij}(x,D_i,D_j)u = 0
\]
Here we use the fact that $-A_{ij}$ admits a bounded $\mathcal{R}^\infty$-calculus [2]. Due to parameter-ellipticity $A_{\Omega}(u) = (A_i + A_j + A_{ij})u$ holds and with the help of the contraction principle of Kahane and the $\mathcal{L}^1$-Lemma 6.5 follows that $M_\lambda$ satisfies condition (5) of the discrete operator-valued Mihlin result.

b) Suitable conditions allow to treat non-constant coefficients and lower order terms of $A_{ij}$ by perturbation arguments. For arbitrary $\alpha \in \mathbb{N}$, we can now copy the proof with $A_{\Omega}$ replaced by $A_i$ and $A_{ij}$ replaced by $A_{ij}$. Here the continuous operator-valued Mihlin result, in particular condition (4), is used.

Corollaries and generalizations
Periodic boundary conditions on $(0,2\pi)^m$ lead to generalized Dirichlet or generalized Neumann boundary conditions on $(0,2\pi)^m$ [3].

Mixed orders for the differential operators $A_i(x, D_i)u$ say $2m_i$ with $m_i \in \mathbb{N}$ for $i = 1,\ldots,m$. Then, parameter-ellipticity of $A_{\Omega}$ has to be replaced by parameter-ellipticity of $A_i$.

For $\mathbb{C} \in \mathbb{C}$ more generally $\mathbb{C}$-periodic boundary conditions $D_{ij}(u)$ where
\[
D_{ij}(u) = 0, D_{ij}(u) = 0
\]
with respect to $(0,2\pi)^m$ can be treated. The particular choice $\alpha \in \{\pm 1\}$ covers mixed periodic and aperiodic boundary conditions.

An application: Chip-cooling
Our main theorem applies to the negative Laplacian in a cubical domain with mixed Dirichlet and Neumann boundary conditions. As the corresponding $\mathcal{R}^\infty$-angle is equal to zero, maximal $\mathcal{L}$-regularity in the cubical domain is established.

This models the cooling of a computer chip or a chip on board with heat pipes on opposite sides.

Source: chipcooler, June 2009, www.chipcooler.com

References