SYMMETRIC NONNEGATIVE FORMS AND SUMS OF SQUARES

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Abstract. We study the asymptotic relationship between nonnegative forms and sums of squares for the case of symmetric forms of degree $2d$, as the number of variables grows. We relate nonnegative symmetric forms to symmetric mean inequalities, valid independent of the number of variables, while sums of squares are approached through the dual cone. In sharp contrast to the non-symmetric case, the difference between symmetric nonnegative forms and sums of squares of degree 4 asymptotically goes to 0. More precisely, given a symmetric quartic, the related symmetric mean inequality holds for all $n \geq 4$, if and only if for all $n \geq 4$, the symmetric quartic can be written as a sum of squares. We conjecture that this is true for arbitrary degree and show that the unlike the non-symmetric case, the difference between symmetric forms and sums of squares degree does not grow arbitrarily large for any degree $2d$.

1. Introduction

Let $\mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_n]$ denote the ring of polynomials in n real variables and $H_{n,k}$ the set of forms of degree k. Certifying that a form $f \in H_{n,2d}$ assumes only nonnegative values is one of the fundamental questions of real algebra. One such possible certificate is a decomposition of f as a sum of squares, i.e., one finds forms $p_1, \ldots p_m \in H_{n,d}$ such that $f = p_1^2 + \ldots + p_m^2$. In 1888 Hilbert gave a beautiful proof showing that in general not all nonnegative forms can be written as a sum of squares. In fact, he showed that the sum of squares property only characterizes nonnegativity in the cases of binary forms, of quadratic forms, and of ternary quartics. In all other cases there exist forms that are nonnegative but do not allow a decomposition as a sum of squares. Despite its elegance, Hilbert's proof was not constructive and it took until the 1950s, when Motzkin provided the first example of such a polynomial.

The SOS decomposition of nonnegative polynomials has been the cornerstone of recent developments in polynomial optimization. Following the ideas of Lasserre and Parrilo, polynomial optimization problems, i.e. the task of finding $f^* = \min f(x)$ for $f \in H_{n,2d}$ can be relaxed and transferred into semidefinite optimization problems. If $f - f^*$ can be written as a sum of squares, these semidefinite relaxations are in fact exact. Hence a better understanding of the difference of sums of squares and nonnegative polynomials is highly desirable.

In [1] the first author added to the work of Hilbert by showing that the quantitative gap between sum of squares and nonnegative polynomials of fixed degree grows infinitely large with the number of variables if the degree is at least 4.

In this article we study the case of forms in n variables of degree $2d$ that are symmetric, i.e., invariant under the action of the symmetric group S_n that permutes the variables. Let $\Sigma_{n,2d}^S$ denote the symmetric forms of degree $2d$ in n variables that admit a decomposition into a sum of squares and $\mathcal{P}_{n,2d}^S$ be the cone of nonnegative symmetric forms. Choi and Lam [3] showed that the following symmetric form of degree 4

$$\sum x_i^2x_j^2 + \sum x_i^2x_jx_k - 4x_1x_2x_3x_4$$

is nonnegative but cannot be written as a sum of squares. Thus one can conclude that $\Sigma_{4,4}^S \neq \mathcal{P}_{4,4}^S$ and therefore even in the case of symmetric polynomials the sum of squares property already fails to characterize nonnegativity in the first case covered by Hilbert's classical result.

This article compares the quantitative gap between symmetric nonnegative polynomials and symmetric sums of squares to the general situation. To this end we consider the family of symmetric power mean inequalities naturally associated to a symmetric polynomial. The study of such symmetric inequalities has a long history (see for example [5]). We show that in general sums of squares do not become asymptotically neglectable compared to symmetric nonnegative forms. In particular in the case of degree 4 we show, that a family of symmetric power mean inequalities is positive for all n if and only if each member can be written as a sum of squares. This implies that the gap between symmetric sums of squares and symmetric nonnegative polynomials vanishes asymptotically in the case of degree 4. For the case of higher degrees, we are only able to show that the gap does not grow infinitely large. We conjecture that the difference between symmetric nonnegative polynomials and symmetric sums of squares asymptotically tends to 0 true for any fixed degree $2d$.

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Contributions: We provide a characterization of the cone of symmetric sums of squares in degree 4 (Theorem 5.2). In particular in Theorem 5.7 we describe the faces which are not faces of the cone of nonnegative polynomials and provide an example (Example 5.8) of a family of symmetric polynomials, which for all numbers of variables lay on a face of the SOS cone but are strictly positive. To compare the asymptotic behavior of sums of squares with symmetric nonnegative polynomials we associate a symmetric form with a symmetric power mean inequality. We characterize the cone of such symmetric inequalities using the half-degree principle. Building on this, we show that a symmetric inequality of degree 4 holds for all n if and only if it can be written as a sum of squares (Theorem 6.3). Further we remark that for a general degree the gap between symmetric nonnegative forms and symmetric sums of squares shrinks when the number of variables is growing and we conjecture that the for all degrees symmetric power mean inequalities can be completely understood via sums of squares.

The article is structured as follows: In the next section we provide a short basic introduction to symmetric functions. In section 3 we investigate properties of the cones of symmetric nonnegative polynomials of degree 2d. Section four establishes some structural results on symmetric sums of squares in degree 4. The final section shows that in degree four the positive power mean inequalities can be characterized by sums of squares.

2. Symmetric polynomials and symmetric functions

For $n \in \mathbb{N}$ let $\mathcal{S}_n$ denote the symmetric group on n letters. A form $f \in H_{n,2d}$ is symmetric, if we have $f(\sigma(x)) = f(x)$ for all $\sigma \in S_n$ and all $x \in \mathbb{R}^n$. Let $H_{n,2d}^S$ be the space of such forms. We work with the family of normalized power sum symmetric polynomials.

Definition 2.1. *For $n \in \mathbb{N}$ define the normalized i-th power sum polynomial to be*

$$p_i := \frac{1}{n}(x_1^i + \ldots + x_n^i).$$

We will write $p_i^{(n)}$ to emphasize the dependence on the number of variables n.

It is classically known that these polynomials are algebraically independent generators of the ring of symmetric polynomials, i.e., every symmetric polynomial f can uniquely be written as $f = g(p_1, \ldots, p_d)$ for some polynomial g.

Definition 2.2. *An ordered sequence of natural numbers $\lambda_1 \geq \lambda_2, \geq \ldots, \geq \lambda_k$ such that $\sum_{i=1}^k \lambda_i = d$ is called a* partition of d. *We write a partition shorthand as $(\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_k^{m_k})$ to indicate that the number λ_i occurred m_i times, and we will denote by $\pi(d)$ the number of partitions of d.*

Since each p_i is a form of degree i we can use a partition λ of $d \in \mathbb{N}$ to define the corresponding form p_λ of degree d.

Definition 2.3. *Let $d \in \mathbb{N}$ and $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash d$. Then $p_\lambda^{(n)} := p_{\lambda_1}^{(n)} \cdot p_{\lambda_2}^{(n)} \cdots p_{\lambda_l}^{(n)}$. Given $\lambda \vdash d$ we also define $m_\lambda := \sum_\beta x_1^{\beta_1} \cdots x_n^{\beta_n}$, wheer β ranges over all distinct permutations of λ.*

Every polynomial $p_\lambda^{(n)}$ for $\lambda \vdash d$ is of degree d. Therefore we get the following that every symmetric polynomial f can be uniquely written as

$$f = \sum_{\lambda \vdash d} c_\lambda p_\lambda^{(n)},$$

where $c_\lambda \in \mathbb{R}$.

This in particular expresses the fact that the dimension of $H_{n,2d}^S$ is independent of n, once $n \geq 2d$, and it makes sense to define symmetric polynomials independent of the number of variables (see for example the construction given in [9]).

Remark 2.4. *For every $n \geq 2d$ the basis of symmetric means $p_\lambda^{(n)}$ realizes the cones $\mathcal{P}_{n,2d}^S$ and $\Sigma_{n,2d}^S$ as convex sets in $\mathbb{R}^{\pi(2d)}$.*

We will be interested in vectors c_λ that define nonnegative forms, and sums of squares regardless of the number of variables n. In order to eliminate dependence on n we consider sequences

$$\mathfrak{p}_\lambda := (p_\lambda^{(2d)}, p_\lambda^{(2d+1)}, \ldots)$$

of symmetric means and study the corresponding linear combinations

$$\mathfrak{f} = \sum_{\lambda \vdash 2d} c_\lambda \mathfrak{p}_\lambda.$$

The set of all such sequences of linear combinations of power means of degree $2d$ will be denoted by $\mathfrak{M}_{2d}$. We say that an element $\mathfrak{f}$ of $\mathfrak{M}_{2d}$ is positive semidefinite (psd), if all functions in the sequence are nonnegative. The question of characterizing the elements of the cone

$$\mathfrak{P}_{2d} := \{\mathfrak{f} \in \mathfrak{M}_{2d} : \ f^{(n)} \in \mathcal{P}_{n,2d}^S \text{ for all } n \geq 2d\}$$

is related to understanding symmetric inequalities such as Newton's inequalities (see [5]). We will compare the positive semidefinite symmetric means to sums of squares. Hence we analogously define

$$\mathfrak{S}_{2d} := \{\mathfrak{f} \in \mathfrak{M}_{2d} \,:\, f^{(n)} \in \Sigma^{S}_{n,2d} \text{ for all } n \geq 2d\}.$$

We main to study the relationship of these cones and show in Section 6 that for degree 4 these cones actually coincide (see Theorem 6.3).

3. The cones of symmetric sums of squares and nonnegative forms

In this section we work out the structure of the symmetric quartics that can be written as sums of squares.

3.1. Symmetric nonnegative polynomials.

The key result that we need in order to characterize the elements of $\mathcal{P}^{S_n}_{n.d}$ is the so-called half degree principle (see [18, 13]): For a natural number $d \in \mathbb{N}$ we define A_d to be the set of all points in $\mathbb{R}^n$ with at most d distinct components

$$A_d := \{x \in \mathbb{R}^n \,:\, |\{x_1, \dots, x_n\}| \leq d\}.$$

The half degree principle says in order to check if a symmetric polynomial of degree $2d$ is nonnegative, it suffices to check its nonnegativity on A_d:

Proposition 3.1 (Half degree principle). *Let $f \in \mathbb{R}[x]^{S_n}$, $2d \in \mathbb{N}$ be the degree of f, and set $k := \max\{2, \lfloor \frac{d}{2} \rfloor\}$. Then f is nonnegative if and only if*

$$f(y) \geq 0 \textit{ for all } y \in A_k.$$

An ordered sequence of k natural numbers $\vartheta := (\vartheta_1, \dots, \vartheta_k)$ such that $\vartheta_1, + \dots, + \vartheta_k = n$ is called a k-partition of n (written $\vartheta \vdash_k n$). Given asymmetric polynomial $f \in \mathbb{R}[x]$ of degree $2k$ in n variables and ϑ a k-partition of n we define $f^{\vartheta} \in \mathbb{R}[t_1, \dots, t_k]$ via

$$f^{\vartheta}(t_1, \dots, t_k) := f(\underbrace{t_1, \dots, t_1}_{\vartheta_1}, \underbrace{t_2, \dots, t_2}_{\vartheta_2}, \dots, \underbrace{t_k, \dots, t_k}_{\vartheta_k}).$$

In particular, for every $l \in \mathbb{N}$ we have $p_l^{\vartheta} = \frac{1}{l}(\vartheta_1 t_1^l + \vartheta_2 t_2^l + \dots + \vartheta_k t_k^l)$. The degree principle then amounts to say that $f \in \mathcal{P}_{n,d}$ if and only if $f^{\vartheta} := \sum\limits_{\lambda \vdash d} c_{\lambda} p_{\lambda}^{\vartheta}(t_1, \dots t_k)$ is nonnegative.

We consider the set

$$K_d := \{k = (k_1, \dots k_d) \in \{\frac{1}{n}, \frac{2}{n}, \dots, 1\}^d \text{ such that } k_1 \leq k_2 \leq \dots \leq k_d \text{ and } k_1 + \dots + k_d = 1\}.$$

Further for any $k \in K_d$ and $\lambda = (\lambda_1, \dots, \lambda_l) \vdash 2d$ we define d-variate forms

$$\Phi^{k}_{p_\lambda} := \prod_{i=1}^{l} (k_1 x_1^{\lambda_i} + k_2 x_2^{\lambda_i} + \dots + k_d x_d^{\lambda_i}).$$

To any $f := \sum_{\lambda \vdash 2d} c_\lambda p_\lambda$ and $k \in K_d$ we now associate the form

$$\Phi^{k}_{f} := \sum_{\lambda \vdash 2d} c_\lambda \Phi^{(n)}_{p_\lambda}$$

Now the following corollary relates the nonnegativity of f with the family of d-variate forms

Corollary 3.2. *Let $f := \sum_{\lambda\vdash 2d} c_\lambda p_\lambda$ be a symmetric form. Then f is nonnegative (positive) if and only if for all $k \in K_d$ the d variate form Φ_f^k is nonnegative (positive).*

3.2. **Invariant sums of squares.** Let $f \in \mathbb{R}[x]$ be a form of degree $2d$, G be a finite group acting linearly on $\mathbb{R}^n$ and assume that f is invariant with respect to G.

As G acts linearly on $\mathbb{R}^n$ also the $\mathbb{R}-$vector space $H_{n,d}$ can be viewed as a G-module and by Maschke's theorem (the reader may consult for example [16] for basics in linear representation theory) there exists a decomposition of the form

$$H_{n,d} \;=\; V^{(1)} \oplus V^{(2)} \oplus \cdots \oplus V^{(h)} \tag{3.1}$$

with $V^{(j)} = W_1^{(j)} \oplus \cdots \oplus W_{\eta_j}^{(j)}$ and $\nu_j := \dim W_i^{(j)}$. Here, the $W_i^{(j)}$ are the so called *irreducible components* and the $V^{(j)}$ are the *isotypic components*, i.e., the direct sum of isomorphic irreducible components. The component with respect to the trivial irreducible representation is the invariant ring $\mathbb{R}[x]^G$. The elements of the other isotypic components are called *semi-invariants.* To any element $f \in H_{n,d}$ we can associate a *symmetrization* by which we mean its image under the following linear map:

Definition 3.3. *For a finite group G the linear map $\mathcal{R}_G : H_{n,d} \to H_{n,d}^G$ which is defined by*

$$\mathcal{R}_G(f) := \frac{1}{|G|}\sum_{\sigma\in G}\sigma(f)$$

is called the Reynolds operator of G. In the case of $G = S_n$ we say that $\mathcal{R}_{S_n}(f)$ is a symmetrization of f and we write $\operatorname{sym}(f)$ *in this case.*

Consider a set of forms $\{f_{1,1},\ldots,f_{1,2},\ldots,f_{m,\eta_m}\}$ such that for fixed j the forms $f_{j,i}$ are (semi)-invariants which generate the distinct irreducible modules decomposing $V^{(j)}$. Further assume that they are chosen in such a way, that for each j and each pair (l,k) there is a G-isomorphism $\rho_{l,k}^{(j)}$ which maps $f_{j,l}$ to $f_{j,k}$.

Let $f_1,\ldots,f_l$ be a (finite) set of polynomials, we will write $\sum \mathbb{R}\{f_1,\ldots,f_l\}^2$ to refer to a sum of squares of the polynomials $f_1,\ldots,f_l$. It has already been observed by Gaterman and Parrilo [6] that invariant sums of squares can be written as sums of squares of semi-invariants using Schur's Lemma. However, in the case of groups all of whose real irreducible representations are also irreducible as complex representations - as for example in the case of S_n, we can deduce even more about the decomposition into sums of squares.

Theorem 3.4. *Let G be a finite group, all of which real irreducible representations $V \subset H_{n,d}$ are absolutely irreducible and f a form of degree $2d$ which is invariant with respect to G. Then if f is a sum of squares, f can be written in the form*

$$f = \mathcal{R}_G\left(\sum \mathbb{R}\left\{f_{1,1},\ldots f_{1,\eta_1}\right\}^2\right) + \mathcal{R}_G\left(\sum \mathbb{R}\left\{f_{2,1},\ldots,f_{2,\eta_2}\right\}^2\right) + \ldots$$

Proof. We assume that $f \in H_{n,2d}$ is a sum of squares and further that f is G-invariant. Since f can be written as a sum of squares there exists a symmetric positive semidefinite bilinear form

$$B : H_{n,d} \times H_{n,d} \to \mathbb{R}$$

which is a Gram-form for f, i.e. for every $x \in \mathbb{R}^n$ we can write $f(x) = B(x^d, x^d)$, where x^d stands for the d-power in the symmetric algebra of $\mathbb{R}^n$. Since f is supposed to be $G-$invariant, we have that $f = \mathcal{R}_G(f)$ and by linearity we can thus assume that B is a G-invariant bilinear form. Now assume a decomposition of $H_{n,2d}$ as in (3.1) and consider the restriction of B to

$$B^{ij} : V^{(i)} \times V^{(j)} \to \mathbb{R} \text{ with } i \neq j.$$

But then for every $v \in V^{(i)}$ the quadratic form B^{ij} defines a linear map $\phi_v : V^{(j)} \to \mathbb{R}$ via $\phi_v(w) := B^{ij}(v,w)$ and so B^{ij} naturally can be seen as an element of $\mathrm{Hom}^G(V^{(i)^*}, V^{(j)})$. Since real representations are self dual we have that $V^{(i)^*}$ and $V^{(j)}$ are not isomorphic and thus by Schur's Lemma we find that $B^{ij}(v,w) = 0$ for all $v \in V^{(i)}$ and $w \in V^{(j)}$. So the isotypic components are orthogonal with respect to B and hence it suffices to look at

$$B^{jj} : V^{(j)} \times V^{(j)} \to \mathbb{R}$$

individually. We have $V^{(j)} := \bigoplus_{k=1}^{l} W_k^{(j)}$, where each $W_k^{(j)}$ is generated by a semi-invariant $f_{j,k}$, i.e. there is a generating set $f_{j,k,1}, \ldots, f_{j,k,\nu_j}$ for every $W_k^{(j)}$ such that the $f_{j,k,i}$ are in the orbit of $f_{j,k}$ under G.

Consider a pair $W_{k_1}^{(j)}, W_{k_2}^{(j)}$, where we allow $k_1 = k_2$. To apply Schur's Lemma we relate the quadratic from B^{jj} to a linear map $\psi_{k_1,k_2}^{(j)} : W_{k_1}^{(j)} \to W_{k_2}^{(j)}$ defined on the generating set $f^{j,k_1,1}, \ldots, f_{j,k_1,\nu_j}$ by

$$\psi_{k_1,k_2}^{(j)}(f_{j,k_1,u}) := \sum_v B^{jj}(f_{j,k_1,u}, f_{j,k_2,v}) f_{j,k_2,v}.$$

Since we assumed the W_k^j to be absolutely irreducible we have by Schur's Lemma

$$\dim(\mathrm{Hom}^G(W_{k_1}^{(j)}, W_{k_2}^{(j)})) = 1$$

and we can conclude that this map is unique up to a scalar multiplication. Therefore it can be represented in the form $\psi_{k_1,k_2}^{(j)} = c_{k_1,k_2} \cdot \rho_{k_1,k_2}$, where ρ_{k_1,k_2} is the $G-$ isomorphism with $\rho_{k_1,k_2}(f_{j,k_1}) = f_{j,k_2}$ as above. It therefore follows that

$$B^{jj}(f_{j,k_1,u}, f_{j,k_2,v}) = \delta_{u,v} c_{k_1,k_2},$$

where $\delta_{u,v}$ denotes the Kronecker Delta. Since we have that up to a positive constant $\mathcal{R}_G(f_k^{(j)}) = f_{j,k,1} + \ldots + f_{j,k,\nu_j}$ we can conclude that f has the indicated decomposition. □

As was remarked already an example for a group all of which irreducible representations are real is the symmetric group. Since this is the case we are interested in, we will provide a useful collection of facts from the representation theory of S_n in the next section.

3.3. **Symmetric sums of squares.** In order to apply the results of the last section we will briefly describe the representation theory of the symmetric group and refer to [8, 15] for more details. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$.then a *Young tableau* for λ consists of l rows, with λ_i entries in the i-th row. Each entry is an element in $\{1, \ldots, n\}$, and each of these numbers occurs exactly once. A *standard Young tableau* is a Young tableau in which all rows and columns are increasing. An element $\sigma \in \mathcal{S}_n$ acts on a Young tableau by replacing each entry by its image under σ. Two Young tableaux t_1 and t_2 are called *row equivalent* if the corresponding rows of the two tableaux contain the same numbers. The classes of equivalent Young tableaux are called *tabloids*, and the class of a tableau t is denoted by $\{t\}$. The action of $\mathcal{S}_n$ gives rise the *permutation module* M^λ *corresponding to* λ which is the $\mathcal{S}_n$-module defined by $M^\lambda = \mathbb{R}\{\{t_1\}, \ldots, \{t_l\}\}$, where $\{t_1\}, \ldots, \{t_l\}$ is a complete list of λ-tabloids.

Let t be a Young tableau for $\lambda \vdash n$, and let C_i be the entries in the i-th column of t. The group $\text{CStab}_t = \mathcal{S}_{C_1} \times \mathcal{S}_{C_2} \times \cdots \times \mathcal{S}_{C_\nu}$ (where $\mathcal{S}_{C_i}$ is the symmetric group on $\mathcal{C}_i$) is called the *column stabilizer* of t.The irreducible representations of the symmetric group $\mathcal{S}_n$ are in 1-1-correspondence with the partitions of n, and they are given by the Specht modules, as explained in the following. For $\lambda \vdash n$, the *polytabloid associated with* t is defined by

$$e_t = \sum_{\sigma \in \text{CStab}_t} \text{sgn}(\sigma)\sigma\{t\}\,.$$

Then for a partition $\lambda \vdash n$, the *Specht module* S^λ is the submodule of the permutation module M^λ spanned by the polytabloids e_t. The dimension of S^λ is given by the number of standard Young tableaux for $\lambda \vdash n$. A generalized *Young tableau* of *shape* λ is a Young tableau T for λ such that the entries are replaced by any n-tuple of natural numbers. The *content* of T is the sequence μ_i such that μ_i is equal to the number of $i's$ in T. A generalized Young tableau is called *semi standard*, if its rows weakly increase and its columns strictly increase.

The decomposition of the module M^λ for a general partition $\lambda \vdash n$ will be of special interest for us. It can be described in a rather combinatorial way as follows:

Proposition 3.5. *For a partition $\mu \vdash n$, the permutation module M^μ can be decomposed as*

$$M^\mu = \bigoplus_{\lambda \trianglerighteq \mu} K_{\lambda\mu} S^\lambda,$$

where for two $\lambda, \mu \vdash n$ the number $K_{\lambda\mu}$ is the so called Kostka number*, which is defined to be the number of semi standard Young tableaux of shape λ and content μ*

Theorem 3.6. *Let $f \in \mathbb{R}[X]$ be a symmetric sum of squares of degree $2d$ then f can be written as $f = \text{sym}(\sigma(p_1^d, \ldots, p_d)+$.*

4. Asymptotics of the cones

To see the connection of the asymptotic behavior of nonnegative symmetric polynomials to the cone $\mathfrak{P}_{2d}$ we establish the following

Proposition 4.1. *Consider the cones $\mathcal{P}_{n,2d}^S$ as convex subsets of $\mathbb{R}^{\pi(d)}$ using the coefficients c_λ of p_λ. Then for every $n \geq 2d$ and $\ell \in \mathbb{N}$ we have*

$$\mathbb{R}^{\pi(2d)} \supset \mathcal{P}_{n,2d}^S \supseteq \mathcal{P}_{\ell \cdot n,2d}^S.$$

Proof. Let $x = (x_1, \ldots, x_n)$ be a point in $\mathbb{R}^n$ and let $\tilde{x}$ be the point in $\mathbb{R}^{\ell \cdot n}$ with each x_i repeated ℓ times. Now we have

$$p_i^{(\ell \cdot n)}(\tilde{x}) = \frac{1}{n\ell}(\ell x_1^i + \ldots + \ell x_n^i) = p_i^{(n)}(x).$$

It follows that $f^{(\ell n)} \in \mathcal{P}_{\ell n,d}^S \Rightarrow f^{(n)} \in \mathcal{P}_{n,d}^S$ which proves the inclusion.

□

Remark 4.2. *We note that the same proof also yields that $\Sigma_{n,2d}^S \supseteq \Sigma_{\ell \cdot n,2d}^S$*

We investigate the asymptotic behavior of symmetric sums of squares and symmetric nonnegative forms and the relation to the cones $\mathfrak{P}_{2d}$ of all sequences $\mathfrak{f} = (f^{(2d)}, f^{(2d+1)}, \ldots)$ of degree $2d$ means that are nonnegative for all n and $\mathfrak{S}_{2d}$ the set of such sequences that can be written as sums of squares.

Proposition 4.3. *Consider the sequences of sets $\mathcal{P}^S_{n,2d}$ and $\Sigma^S_{n,2d}$ (with $n \geq 2d$), realized as subsets of $\mathbb{R}^{\pi(2d)}$ by the coefficients c_λ of p_λ. Then*

$$\mathfrak{P}_{2d} = \lim_{n\to\infty} \mathcal{P}^S_{n,2d} = \bigcap_{n=2d}^{\infty} \mathcal{P}^S_{n,2d},$$

$$\mathfrak{S}_{2d} = \lim_{n\to\infty} \Sigma^S_{n,2d} = \bigcap_{n=2d}^{\infty} \Sigma^S_{n,2d}.$$

Proof. We prove the statement for the cones $\mathcal{P}^S_{n,2d}$. The proof for sums of squares is identical. For a nested sequence of sets S_k with $S_k \supseteq S_{k+1}$ the limit is equal to the intersection $\cap_{k=1}^{\infty} S_k$. The sequence $\mathcal{P}^S_{n,2d}$ is not nested, however, by Proposition 4.1 we have that for every $m \in \mathbb{N}$ the inclusion $\mathcal{P}^S_{m,2d} \supseteq \mathcal{P}^S_{\ell\cdot m,2d}$ holds for any $\ell \in \mathbb{N}$. This in turn implies, that every sequence a_i of natural numbers satisfying $a_1 \geq 2d$ and $a_i \,|\, a_{i+1}$ for all i gives rise to a nested (sub)-sequence $\mathcal{P}^S_{a_i,2d}$. Since for each of these sequences the limit is well defined, it suffices to show that all the limits coincide. To prove this claim by contradiction assume that there are two such sequences a_i and b_j such that $\cap_{i=1}^{\infty}\mathcal{P}^S_{a_i,2d} \neq \cap_{j=1}^{\infty}\mathcal{P}^S_{b_j,2d}$. Without loss of generality there exists $s \in \mathbb{R}^{\pi(d)}$ and $k \in \mathbb{N}$ such that $s \in \cap_{i=1}^{\infty}\mathcal{P}^S_{a_i,2d}$ and $s \notin \mathcal{P}^S_{b_j,2d}$ for all $j \geq k$. However, we see that for any a_i, $s \in \mathcal{P}^S_{a_i,2d}$ implies that $s \in \mathcal{P}^S_{b_k\cdot a_i,2d} \subseteq \mathcal{P}_{b_k,2d}$ which gives the desired contradiction. □

Note that in particular, Proposition 4.3 implies that the $\mathfrak{P}_{2d}$ and $\mathfrak{S}_{2d}$ are closed, pointed convex cones. A priori it is not clear that the cones $\mathfrak{S}_{2d}$ and $\mathfrak{P}_{2d}$ are full dimensional. We first establish this via the following Lemma.

Lemma 4.4. *For every $d, n \in \mathbb{N}$ the cone $\Sigma^S_{n.2d}$ contains the cone $\Sigma'_{n,2d}$ of forms*

$$\sum_{\lambda_1,\lambda_2 \vdash d} \alpha_{\lambda_1,\lambda_2} p^{(n)}_{\lambda_1} p^{(n)}_{\lambda_2} + \sum_{i,j=1}^{d} \sum_{\substack{\mu_1 \vdash d-i \\ \mu_2 \vdash d-j}} \beta^{i,j}_{\mu_1,\mu_2} (p^{(n)}_{(i+j} p^{(n)}_{(\mu_1)} p^{(n)}_{(\mu_2)} - p^{(n)}_i p_j p_{\mu_1} p_{\mu_2}), \tag{4.1}$$

where the coefficients $\{\alpha_{\lambda_i,\lambda_j}\}$ and $\{\beta^{i,j}_{\mu_1,\mu_2}\}$ come from positive definite quadratic forms. This cone has full dimension in $\mathbb{R}^{\pi(2d)}$.

Proof. The set $\{p_\lambda, \lambda \vdash d\}$ is a basis of the space of symmetric forms of degree d. Hence any sum of squares of symmetric polynomials of degree d can be written in the form

$$\sum_{\lambda_1,\lambda_2 \vdash d} \alpha_{\lambda_1,\lambda_2} p_{\lambda_1} p_{\lambda_2}.$$

Further for every $i \in \{1, \ldots, d\}$ we consider the forms

$$\left\{ \frac{1}{n}(x_1^i - x_2^i), \frac{1}{n}(x_1^i - x_3^i), \ldots, \frac{1}{n}(x_{n-1}^i - x_n^i) \right\}.$$

and

$$\left\{ \frac{1}{n}(x_1^i - x_2^i) \cdot p^{(n)}_\mu, \frac{1}{n}(x_1^i - x_3^i) \cdot p^{(n)}_\mu, \ldots, \frac{1}{n}(x_{n-1}^i - x_n^i) \cdot p^{(n)}_\mu \right\} \subset \mathbb{R}[x]_d.$$

Given two distinct partitions $\mu_1, \mu_2 \vdash d$ we obviously have that the two associated invariant subspaces of $\mathbb{R}[x]_d$ are distinct, by inspecting the monomials involved. Finally we calculate that for $i, j \in \{1, \ldots, n\}$ and $\mu_1, \mu_2 \vdash d$ we have

$$\sum_{v<w} \frac{1}{n}(x_v^i - x_w^i)p_{\mu_1}^{(n)} \frac{1}{n}(x_v^j - x_w^j)p_{\mu_2}^{(n)} = (p_{i+j}^{(n)} p_{\mu_1}^{(n)} p_{\mu_2}^{(n)} - p_i^{(n)} p_j^{(n)} p_{\mu_1}^{(n)} p_{\mu_2}^{(n)}).$$

In order to show, that the cone we generated in this way is full dimensional we argue as follows: We distinguish two types of partitions $\lambda \vdash 2d$: Namely those with $l_1 \leq d$ and those with $l_1 > d$. As the first sort of partitions will only involve numbers $\leq d$ we can realize each such partition of $2d$ by concatenate two partitions of d and order the resulting sequence of numbers. As $l_1 > d$ implies that $d > l_2 \geq l_3 \geq \ldots \geq l_k$ we can realize each partition of the second group by specifying l_1 and then concatenate any partition of $2d - l_1$. Taking this thoughts into account we deduce that the pairwise products of the polynomials $p_{\tilde{\lambda}}^{(n)}$, $\tilde{\lambda} \vdash d$ in fact generate all polynomials $p_{\lambda}^{(n)}$, with $\lambda \vdash 2d$ comming from the first group. Similarly chose i, j such that $i + j > d$ then $(p_{i+j}^{(n)} p_{\mu_1}^{(n)} p_{\mu_2}^{(n)} - p_i^{(n)} p_j^{(n)} p_{\mu_1}^{(n)} p_{\mu_2}^{(n)})$ will generate all polynomials $p_{\lambda}^{(n)}$ with $\lambda \vdash 2d$ coming from the second group. Therefore the space spanned by all polynomials as in (5.1) has the right dimension. □

Theorem 4.5. *For any $d \in \mathbb{N}$, the cones $\mathfrak{S}_{2d}$ and $\mathfrak{P}_{2d}$ are full dimensional.*

Proof. From the above Proposition we know that for any finite n the cone ${\Sigma_{n,2d}^S}'$ is full dimensional. Therefore we define

$$\mathfrak{S}'_{2d} := \bigcap_{n \geq 2d} {\Sigma_{n,2d}^S}'$$

Since the description of $\Sigma_{n,2d}^S$ does not change with n, we have that the full dimensionality of each of these cones directly carries over to $\mathfrak{S}'_{2d}$. But then the claim follows from the following chain of inclusions:

$$\mathfrak{S}'_{2d} \subset \mathfrak{S}_{2d} \subseteq \mathfrak{P}_{2d}.$$

□

Let $d \in \mathbb{N}$ and consider $I_n \subset H_{n,2d} := \{f^{(n)} := \sum_{\lambda \vdash 2d} c_\lambda p_\lambda^{(n)} \, : \, \sum_{\lambda \vdash 2d} c_\lambda = 1\}$. Since the section of I_n with $\mathcal{P}_{n,2d}$ and $\Sigma_{n,2d}$ is compact it makes sense to view the ratio of the resulting volumes as measure of the difference of the two comes. To this end we denote by $v_{\mathcal{P}(n,2d)} := \mathrm{vol}(\mathcal{P}_{n,2d})$ and $v_{\Sigma(n,2d)} := \mathrm{vol}(\Sigma_{n,2d})$. By combining Theorem 4.5 with Proposition 4.3 we get

Corollary 4.6. $\lim_{n \to \infty} \frac{v_{\Sigma(n,2d)}}{v_{\mathcal{P}(n,2d)}} > 0$.

In view of the results in [1] the above Corollary shows that asymptotically the situation with symmetric sums of squares is much nicer. In the remain of the paper we aim to prove that the situation is particularly nice in the case of quartics.

5. Symmetric sums of squares in degree 4

We now turn to use the above described representation theory of the symmetric group in order to explicitly characterize symmetric sums of squares in degree 4. The first step is to understand the decomposition of $H_{n,2}$ into S_n-modules.

Lemma 5.1. *Let $n \geq 4$. Then we have*

$$H_{n,2} \cong 2S^{(n)} \oplus 2S^{(n-1,1)} \oplus S^{(n-2,2)}$$

as S_n-modules.

Proof. Let $n \geq 4$. We consider exponent vectors $\alpha_1 := (2, 0, \dots, 0)$, $\alpha_2 := (1, 1, 0, \dots, 0) \in \mathbb{N}^n$. To each such exponent vector let $\mathbb{R}\{x^{\alpha_i}\}$ denote the S_n-module generated by x^{α_i}. In this way we construct a decomposition of $H_{n,2}$ as $H_{n,2} \cong \mathbb{R}\{x^{\alpha_1}\} \oplus \mathbb{R}\{x^{\alpha_2}\}$. Now since the stabilizer of x^{α_1} is isomorphic to $S_1 \times S_{n-1}$, we find that $\mathbb{R}\{x^{\alpha_1}\} \cong M^{(n-1,1)}$, similarly we find $\mathbb{R}\{x^{\alpha_2}\} \cong M^{(n-2,2)}$. Combining this with Proposition 3.5 we deduce that $\mathbb{R}\{x^{\alpha_1}\} \cong S^{(n)} \oplus S^{(n-1,1)}$ and $\mathbb{R}\{x^{\alpha_2}\} \cong S^{(n)} \oplus S^{(n-1,1)} \oplus S^{(n-2,2)}$ which puts together the desired decomposition. □

Using this decomposition in combination with Theorem 3.4 we can calculate the decomposition of a general symmetric quartic into a sum of squares a follows.

Theorem 5.2. *Let $f \in H_{n,4}$ be symmetric and $n \geq 4$. Then f is a sum of squares, if and only if f can be written in the form*

$$\begin{aligned} f &= \alpha_1 p^{(n)}_{(1^4)} + 2\alpha_2 p^{(n)}_{(2,1^2)} + \alpha_3 p^{(n)}_{(2^2)} + \beta_1 \left(p^{(n)}_{(2,1^2)} - p^{(n)}_{(1^4)}\right) + 2\beta_2 \left(p^{(n)}_{(3,1)} - p^{(n)}_{(2,1^2)}\right) + \beta_3 \left(p^{(n)}_{(1^4)} - p^{(n)}_{(2^2)}\right) \\ &\quad + \gamma \left(\tfrac{n^2}{4} p^{(n)}_{1^4} - n p^{(n)}_{(21^2)} + p^{(n)}_{(3,1)} + \tfrac{n-1}{4} p^{(n)}_{(2^2)} - \tfrac{1}{4} p^{(n)}_{4}\right) \end{aligned}$$

such that $\alpha_1\alpha_3 - \alpha_2^2 \geq 0$, $\beta_1\beta_3 - \beta_2^2 \geq 0$, $\gamma \geq 0$.

Proof. In view of Theorem 3.4 we can examine the irreducible components as given in Lemma 5.1 separately. The invariant polynomials of degree 2 are spanned by $p^{(n)}_{(1,1)}$ and $p^{(n)}_{(2)}$ and thus the set

$$\{\alpha_1 p^{(n)}_{(1,1,1,1)} + 2\alpha_2 p^{(n)}_{(2,1,1)} + \alpha_3 p^{(n)}_{(2,2)}\}, \tag{5.1}$$

with $\alpha_1\alpha_3 - \alpha_2^2 \geq 0$ corresponds to the possible sums of squares of symmetric quadratic forms. As indicated in Lemma 5.1 we need to identify two district copies of $S^{(n-1,1)}$ as well as one copy of $S^{(n-2,2)}$ as submodules of $H_{n,2}$. We start with the two copies of $S^{(n-1,1)}$:

Let $\{t\}$ be a $(n-1,1)-$ tabloid denote the unique integer in the second row of $\{t\}$ by i. Then the map $\{t\} \mapsto x_i$ is a S_n-isomorphism between $M^{(n-1,1)}$ and $H_{n,1}$. Therefore it is easy to see that the S_n-module spanned by all linear forms $x_i - x_j$ with $i \neq j$ is isomorphic to $S^{(n-1,1)}$. We pick one of these polynomials, namely we define

$$s_{(n-1,1)} := x_1 - x_2$$

To realize the irreducible module $S^{(n-1,1)}$ as a submodule of $H_{n,2}$, we use the two polynomials

$$f_1^{(n-1,1)} := \frac{1}{n} s_{(n-1,1)} p_1^{(n)} \text{ and } f_2^{(n-1,1)} := \frac{1}{n} s_{(n-1,1)}(x_1 + x_2) = \frac{1}{n}(x_1^2 - x_2^2).$$

We have that the stabilizer of both $f_1^{(n-1,1)}$ and $f_2^{(n-1,1)}$ equals the the stabilizer of $s_{(n-1,1)}$. Further we have $\sigma(f_1^{(n-1,1)}) = -f_1^{(n-1,1)}$ and $\sigma(f_2^{(n-1,1)}) = -f_2^{(n-1,1)}$ for the transposition $\sigma \in S_2$. Thefore we can conclude that $f_1^{(n-1,1)}$ and $f_2^{(n-1,1)}$ both span an S_n-module isomorphic to $S^{(n-1,1)}$. Furthermore a comparison of the monomials involved shows that both modules are distinct. Therefore the sums of squares of semi-invariants of type $(n-1,1)$ can be written as

$$\mathrm{sym}(\beta_1(f_1^{(n-1,1)})^2+2\beta_2 f_1^{(n-1,1)}f_2^{(n-1,1)}+\beta_3(f_2^{(n-1,1)})^2)=$$
$$\frac{1}{n^2}\sum_{i<j}\beta_1(x_i-x_j)^2p_{(1^2)}^{(n)}+2\beta_2(x_i-x_j)p_1^{(n)}(x_i^2-x_j^2)+\beta_3(x_i^2-x_j^2),$$

which gives us exactly the forms

$$\beta_1\left(p_{(2,1^2)}^{(n)}-p_{(1^4)}^{(n)}\right)+2\beta_2\left(p_{(3,1)}^{(n)}-p_{(2,1^2)}^{(n)}\right)+\beta_3\left(p_{(1^4)}^{(n)}-p_{(2^2)}^{(n)}\right). \tag{5.2}$$

Finally also the sums of squares of semi invariants of type $(n-2,2)$ need to be considered. Using similar arguments like above we deduce that the polynomial

$$f_1^{(n-2,2)}:=(x_1-x_2)(x_3-x_4)$$

spans an S_n-module which is isomorphic to the Specht-Module $S^{(n-2,2)}$. Since by Lemma 5.1 there is only one occurrence of $S^{(n-2,2)}$ and as $f_1^{(n-2,2)}\in H_{n,2}$, we need to calculate $\mathrm{sym}((f_1^{(n-2,2)})^2)$. We have

$$(f_1^{(n-2,2)})^2:=x_1^2x_3^2+x_1^2x_4^2+x_2^2x_3^2+x_2^2x_4^2-2(x_2^2x_3x_4+x_1^2x_3x_4+x_1x_2x_3^2+x_1x_2x_4^2)+4x_1x_2x_3x_4.$$

Since the stabilizer of a monomial of the form $x_i^2x_j^2$ with $j\neq j$ is of the from $S_2\times S_{n-2}$, we have that

$$\sum_{\sigma\in S_n}\sigma(x_i^2x_j^2)=2(n-2)!M_{2,2},$$

where $M_{2,2}$ is a monomial symmetric function, i.e., the sum over all possible monomials $x_i^2x_j^2$ with $i\neq j$. Following the same idea we get that for each triplet (i,j,k) with $i\neq j\neq k$ we have

$$\sum_{\sigma\in S_n}\sigma(x_i^2x_jx_k)=2(n-3)!M_{2,1,1},$$

where $M_{2,1,1}$ denotes the corresponding monomial symmetric function. Finally we also have that for every 4-tuple (i,j,k,l) we have

$$\sum_{\sigma\in S_n}\sigma(x_ix_jx_kx_l)=4!(n-4)!.$$

Putting all this together we get

$$\sum_{\sigma\in S_n}\sigma(s_{(n-2,2)^2})=4(2(n-2)!M_{2,2})-2\cdot 4(n-3)!2M_{2,1,1}+4(4!(n-4)!M_{1,1,1,1}).$$

Since we only need to care about $\mathrm{sym}(s^2_{(n-2,2)})$ up to a positive factor, we can simplify this to

$$(n-2)(n-3)M_{2,2}-2(n-3)M_{2,1,1}+12M_{1,1,1,1},$$

which when expressed in the normalized power sum bases yields

$$\frac{n^2}{4}p_{1^4}^{(n)}-np_{(21^2)}^{(n)}+p_{(3,1)}^{(n)}+\frac{n-1}{4}p_{(2^2)}^{(n)}-\frac{1}{4}p_4^{(n)}. \tag{5.3}$$

And we can conclude by combining (5.1), (5.2), (5.3) which yields the indicated form of any symmetric sum of squares of degree 4. □

5.1. **The boundary of $\Sigma^S_{n,4}$.** In the following we analyze the difference of the boundaries of $\Sigma^S_{n,4}$ and $\mathcal{P}^S_{n,4}$, in particular we characterize the facets of the cone $\Sigma^S_{n,4}$ which are not facets of the cone $\mathcal{P}^S_{n,4}$. The existence of such facets is equivalent to $\Sigma^S_{n,4} \neq \mathcal{P}^S_{n,4}$. The key to this examination of the boundary is the dual correspondence: Recall, that for a given convex cone $K \subset \mathbb{R}^n$ the dual cone K^* is defined as

$$K^* := \{\ell \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}) \,:\, \ell(x) \geq 0 \,\text{for all } x \in K\}.$$

The dual correspondence yields that any facet F of K, i.e. every maximal face of K, is given by an extreme ray in K^*, i.e.

$$F = \{x \in K \,:\, \text{such that } \ell(x) = 0\},$$

where $\ell \in K^*$ is an extreme ray of K^*.

Let $S_{n,d}$ be the vector space of real quadratic forms on $H_{n,d}$. An element $Q \in S_{n,d}$ is said to be S_n-invariant if $Q(f) = Q(\sigma(f))$ for all $\sigma \in S_n$. We will denote by $S^{S_n}_{n,d}$ the space of S_n-invariant quadratic forms on $H_{n,d}$. Further we can identify a linear functional $\ell \in H^S_{n,2d}$ with its quadratic form Q_ℓ defined by $Q_\ell(f) = \ell(\mathrm{sym}(f^2))$.

Let $S^{S_n+}_{n,d}$ be the cone of positive semidefinite forms in $S^{S_n}_{n,d}$, i.e.,

$$S^{S_n+}_{n,d} := \{Q \in S^{S_n}_{n,d} \,:\, Q(f) \geq 0 \text{ for all } f \in H_{n,d}\}.$$

The following Lemma is just the dual version of Theorem 3.4, relates elements $\ell \in (\Sigma^S_{n,2d})^*$ to quadratic forms in $S^{S_n+}_{n,d}$.

Lemma 5.3. *For $\lambda \vdash n$ let η_λ denote the number of irreducible components of type λ occurring in $H_{n,d}$ and $\{f_1^{(n)}, \ldots, f^{(n)}_{\eta_{(n)}}, \ldots, f^{(1^n)}_{\eta_{(1^n)}}\}$ a set of semi-invariants as in Theorem 3.4. Then $\ell \in (\Sigma^S_{n,d})^*$ if and only if the bilinear forms*

$$M_\lambda(f_i^\lambda, f_j^\lambda) = \ell(\mathrm{sym}(f_j^\lambda \cdot f_i^\lambda)$$

are in $S^{S_n+}_{n,d}$

In order to use the dual correspondence we want to characterize the extreme rays of $(\Sigma^S_{n,2d})^*$. Since the above lemma identifies it with a linear section of the cone of positive semidefinite forms the following general result will be at our help: Let S be the vector space of quadratic forms on a real vector space V and S^+ be the cone of positive semidefinite forms on S. Then the following lemma is from [11], (Corollary 4):

Lemma 5.4. *Let L be a linear subspace of S and let K be the section of S^+ with L, i.e.,*

$$K := S^+ \cap L.$$

Then if a quadratic form Q spans an extreme ray of K, its kernel $\mathrm{Ker}\, Q$ is maximal for all forms in L, i.e., if $\mathrm{Ker}\, Q \subseteq \mathrm{Ker}\, P$ we must have $P = \lambda Q$ for some $\lambda \in \mathbb{R}$.

In order to examine the kernels of quadratic forms we use the following construction. Let $W \subset H_{n,d}$ be any linear subspace then we define

$$W_S^{<2>} := \{h \in H^S_{n,2d} \,:\, h = \mathrm{sym}(\sum f_i g_i) \;\text{ with } f_i \in W \text{ and } g_i \in H_{n,d}\}.$$

With the notations from above the following straightforward Proposition can be used to characterize the elements in the kernels of the qudratic forms we consider.

Proposition 5.5. *Let Q_ℓ be a quadratic form in $(\Sigma^S_{n,2d})^*$ and let $W_\ell \subset H_{n,d}$ be the kernel of Q_ℓ. Then $p \in W_\ell$ if and only if $\ell(\operatorname{sym}(pq)) = 0$ for all $q \in H_{n,d}$. In particular $\ell(f) = 0$ for all $f \in W_{l}{}_S^{<2>}$*

In the following Lemma we examine which are the possibile kernels corresponding to extreme rays.

Lemma 5.6. *Suppose that the quadratic form Q spans an extreme ray of $(\Sigma^{S_n}_{4,n})^*$. Then either*

$$\begin{aligned} \operatorname{Ker} Q &\simeq S^{(n)} \oplus S^{(n-1,1)} \quad or \\ \operatorname{Ker} Q &\simeq S^{(n-1,1)} \oplus S^{(n-2,2)} \end{aligned}$$

Proof. Since Q is supposed to be S_n-invariant it is clear, that $\operatorname{Ker} Q \subseteq H_{n,2}$ is an S_n-module. Hence from Lemma 5.1 we get that $\operatorname{Ker} Q$ decomposes in the from

$$\operatorname{Ker} Q \simeq a \cdot S^{(n)} \bigoplus b \cdot S^{(n-1,1)} \bigoplus c \cdot S^{(n-2,2)},$$

where $a, b \in \{0, 1, 2\}$ and $c \in \{0, 1\}$. In oder to prove the above Lemma we have to carefully examine all other possible combinations given by the above equation. As above let W denote the kernel of Q. We first show, that $a = 2$ is impossible: Indeed, in the case $a = 2$ we would have $p_2 \in W$ which implies $p_2^2 \in W_S^{<2>}$, which clearly is impossible since p_2^2 is not on the boundary of $\Sigma^{S_n}_{n,4}$. Since we are looking for extreme rays the kernel W of Q must be maximal and not trivial. As we have $\dim H^{S_n}_{n,4} = 5$ this implies $\dim W_S^{<2>} \leq 4$. But this excludes already the cases $b = 2$ and $a = b = c = 1$ since in both cases we would get $\dim W_S^{<2>} = 5$. However, in the cases $a = b = 1, c = 0$ and $a = 0, b = 1, c = 1$ we have $\dim W_S^{<2>} = 4$. Finally, observe that in the other cases W can not be maximal and we can conclude that $a = b = 1, c = 0$ and $a = 0, b = 1, c = 1$ are the only relevant instances. □

The above description allows us to explicitly characterize those symmetric degree 4 sums of squares that are positive and on the boundary of $\Sigma_{n,4}$.

Theorem 5.7. *Let $n \geq 4$ and $f \in H_{n,4}$ be symmetric and positive. Then $f^{(n)}$ is on the boundary of $\Sigma^S_{4,n}$ if and only if it can be written as*

$$a^2 p^{(n)}_{(4)} + 2ab p^{(n)}_{(31)} + (c^2 - a^2) p^{(n)}_{2^2} + (2cd + b^2 - 2ab) p^{(n)}_{(2,1^2)} + (d^2 - b^2) p^{(n)}_{(1^4)},$$

such that the coefficients a, b, c, d satisfy $a \neq 0, c \neq 0$, and the following 5 inequalities :

$$\begin{aligned} &0 \leq -\frac{1}{a}(c+d)(ad-bc), \qquad 0 \leq -(ad-bc)(ac+ad+bc), \\ &0 \leq \frac{1}{ac}(-ad+bd+bc+ac), \quad 0 \leq -(c+d)(abc+b^2c-a^2d+a^2c), \\ &0 \leq \frac{n^2}{4} + \frac{d}{c} + \frac{d^2}{c^2} - \frac{n-1}{4} + \frac{da-db-bc}{ca} - \frac{-b^2cd - b^2c^2 + a^2d^2}{4c^2a^2} \end{aligned} \tag{5.4}$$

Proof. Suppose f is a form on the boundary of $\Sigma^S_{4,n}$. Then there is an $0 \neq \ell \in (\Sigma^S_{4,n})^*$ with $\ell(f) = 0$, where we can assume that ℓ forms an extreme ray of $(\Sigma^S_{4,n})^*$. Let $W_\ell \subset H_{n,2}$ denote the kernel of ℓ. From Lemma 5.6 we deduce the two possible decompositions of W_ℓ in terms of irreducible components. However, if $W_\ell \simeq S^{(n-1,1)} \oplus S^{(n-2,2)}$ then all $p \in W_\ell$ will have a common zero at $x = (1, 1, 1, \dots, 1)$. Therefore the precondition that f is positive excludes this case which implies that

$$W_\ell \simeq S^{(n)} \oplus S^{(n-1,1)}. \tag{5.5}$$

Let $q \in H_{n,2}$. By Proposition 5.5 we have

$$q \in W_\ell \text{ if and only if } \ell(pq) = 0 \text{ for all } p \in H_{n,2}.$$

In view of (5.5) we can assume that W_ℓ is generated by two polynomials

$$q_1 := (\alpha p_1^2 + \beta p_2) \text{ and } q_2 := \frac{\gamma}{n}(x_1 - x_2)p_1 + \frac{\delta}{n}(x_1^2 - x_2^2).$$

Now for any form r in the kernel of Q_ℓ we must have $\ell(\mathrm{sym}(r \cdot q)) = 0$ for all $q \in H_{n,2}$. This condition yields a system of linear equations: We denote $y_\lambda = \ell(p_\lambda)$. Further Schur's lemma implies that $\ell(q_1 \cdot r) \neq 0$ only if r is invariant and $\ell(q_2 \cdot r) \neq 0$ only if r is in the isotopic component $(n-1, 1)$. Therefore the linear functional ℓ is a solution to the following system of equations:

$$\begin{aligned}
0 &= \ell(\mathrm{sym}(q_2 \cdot \frac{1}{n}(x_1^2 - x_2^2))) = a \cdot y_{(4)} - a \cdot y_{(2,2)} + b \cdot y_{(3,1)} - b \cdot y_{(2,1,1)} \\
0 &= \ell(\mathrm{sym}(q_2 \cdot \frac{1}{n}(x_1 - x_2)p_1)) = a \cdot y_{(3,1)} - a \cdot y_{(2,1^2)} + b \cdot y_{(2,1^2)} - b \cdot y_{(1^4)} \\
0 &= \ell(\mathrm{sym}(q_1 \cdot p_2)) = c \cdot y_{(2^2)} + d \cdot y_{(2,1^2)} \\
0 &= \ell(\mathrm{sym}(q_1 \cdot p_1^2)) = c \cdot y_{(2,1^2)} + d \cdot y_{(1^4)}
\end{aligned}$$

in the unknown values of the linear form $\ell \in (H^S_{n,2})^*$ at the various basis elements. By analyzing the above system one sees that $c = 0$ implies $d = 0$. This however is not compatible with the positivity condition on f: Indeed, in this case W_ℓ would be spanned by q_2 and then all elements of W_ℓ would vanish at $(1, 1, \dots, 1)$. But as this implies that also f has a zero at $(1, 1, \dots, 1)$ and so we can conclude that $c \neq 0$. In the case of $a = 0$ we find that any solution of the above system will have

$$0 = y_{(2^2)} = y_{(3,1)} = y_{(2,1^2)} = y_{(1^4)}.$$

In the case $a \neq 0$ we find that the solution is up to a common multiple

$$y_{(4)} = \frac{-b^2cd - b^2c^2 + a^2d^2}{c^2a^2},\ y_{(2^2)} = -\frac{da - db - bc}{ca},\ y_{(3,1)} = \frac{d^2}{c^2},\ y_{(2,1^2)} = -\frac{d}{c},\ y_{(1^4)} = 1.$$

But since we want that the form $\ell \in (\Sigma^S_{n,d})^*$ we must also take into account that the corresponding quadratic form Q_ℓ has to be positive definite. By Lemma 5.3 this is equivalent to checking that each of the matrices

$$M_{(n)} := \begin{pmatrix} y_{(2^2)} & y_{(2,1^2)} \\ y_{(2,1^2)} & y_{(1^4)} \end{pmatrix}, M_{(n-1,1)} := \begin{pmatrix} y_{(4)} - y_{(2^2)} & y_{(3,1)} - y_{(2,1^2)} \\ y_{(3,1)} - y_{(2,1^2)} & y_{(2,1^2)} - y_{(1^4)} \end{pmatrix} \text{ is positive semidefinite,}$$

$$\text{and } M_{(n-2,2)} := \left(\tfrac{n^2}{4} y_{(1^4)} - y_{(21^2)} + y_{(3,1)} + \tfrac{n-1}{4} y_{(2^2)} - \tfrac{1}{4} y_{(4)} \right) \geq 0.$$

Now in the case $a = 0$ these conditions simplify to $y_{(4)} \geq 0$ and $-\frac{1}{4} y_{(4)} \geq 0$ which implies $y_{(4)} = 0$. Therefore the only $\ell \in (\Sigma^S_{n,2d})^*$ that vanishes on a polynomial as above with $a = 0$ has to be the zero map. Therefore we can need to have $a \neq 0$.

If $a \neq 0$ we get the following matrices

$$M_{(n)} := \frac{1}{ac}\begin{pmatrix} -ad+bd+bc & -ad \\ -ad & ac \end{pmatrix},$$
$$M_{(n-1,1)} := -\frac{c+d}{a^2c^2}\begin{pmatrix} abc+b^2c-a^2d & -a^2d \\ -a^2d & a^2c \end{pmatrix},$$
$$\text{and } M_{(n-2,2)} := \left(\tfrac{n^2}{4}+\tfrac{d}{c}+\tfrac{d^2}{c^2}-\tfrac{n-1}{4}+\tfrac{da-db-bc}{ca}-\tfrac{-b^2cd-b^2c^2+a^2d^2}{4c^2a^2}\right)$$

We calculate that $\det M_{(n)} = -\frac{(c+d)(ad-bc)}{ac^2}$ and $\det M_{(n-1,1)} = -\frac{(c+d)^2(ad-bc)(ac+ad+bc)}{a^2c^4}$. Since a 2×2 matrix is positive semidefinite if and only if both its determinant and its trace are nonnegative we can conclude the equations in (5.4) and thus the statement of the theorem. □

Note that although the first symmetric counter example by Choi and Lam in four variables gives $\Sigma^S_{4,4} \neq \mathcal{P}^S_{4,4}$ it is not clear that this already implies that for all n we have this inequality in the quartics case. However, the following example shows that this is the case, since it produces a sequence of strictly positive symmetric quartics that lie on the boundary of $\Sigma^S_{n,4}$ for all n as a witness for the strict inclusion.

Example 5.8. *Consider family of polynomials*

$$f^{(n)} := a^2p^{(n)}_{(4)} + 2abp^{(n)}_{(31)} + (c^2-a^2)p^{(n)}_{2^2} + (2cd+b^2-2ab)p^{(n)}_{(2,1^2)} + (d^2-b^2)p^{(n)}_{(1^4)},$$

where we set $a=1, b=-\frac{3}{2}, c=1$ and $d=-\frac{5}{4}$. Further consider the linear functional $\ell \in H^*_{n,4}$ with

$$\ell\left(p^{(n)}_{(4)}\right) = \frac{17}{8},\ \ell\left(p^{(n)}_{(2^2)}\right) = \frac{13}{8},\ \ell\left(p^{(n)}_{(3,1)}\right) = \frac{25}{16},\ \ell\left(p^{(n)}_{(2,1^2)}\right) = \frac{5}{4},\ \ell\left(p^{(n)}_{(1^4)}\right) = 1.$$

Then we have $\ell(f^{(n)}) = 0$. In addition the corresponding matrices become

$$M_{(n)} := \begin{pmatrix} \frac{13}{8} & \frac{5}{4} \\ \frac{5}{4} & 1 \end{pmatrix}, M_{(n-1,1)} := \begin{pmatrix} \frac{1}{2} & \frac{5}{16} \\ \frac{5}{16} & \frac{1}{4} \end{pmatrix},\ \text{and } M_{(n-2,2)} := \left(\frac{1}{2}n-\frac{1}{4}\right)^2 - \frac{53}{32}.$$

They are all positive semidefinite and therefore we have $\ell \in (\Sigma^S_{n,4})^*$. This implies that $f^{(n)} \in \partial\Sigma^S_{n,4}$.

*Now we want to argue that for any $n \in \mathbb{N}$ we will have $f^{(n)} > 0$. By Corollary **??** it follows that $f^{(n)}$ has a zero, if and only if there exists $k \in \{\frac{1}{n}, \ldots, \frac{n-1}{n}\}$ such that the bivariate form*

$$\Phi^k_f(x,y)$$

has a zero $\neq (0,0)$. By dehomogenization this is equivalent to the statement that for one $k \in \{\frac{1}{n}, \ldots, \frac{n-1}{n}\}$ the univariate polynomial

$$f_k(t) := \Phi^k_f(t,1) = kx^4 + 1 - k - 3\,(kx+1-k)\left(kx^3+1-k\right) + \frac{11}{4}\,(kx+1-k)^2\left(kx^2+1-k\right) - \frac{11}{16}\,(kx+1-k)^4$$

has a real zero. Since $f^{(n)}$ is a sum of squares and therefore positive definite we have by construction that also $f_k(t) \geq 0$. Therefore the real roots of $f_k(t)$ will have even multiplicity. This *implies that $f_k(t)$ has a real root only if its discriminant $\delta(f_k)$ - viewed as polynomial in the parameter k - has a root in the admissible range for k. We calculate*

$$\delta(f_k) := -\frac{1}{256}\left(16 - 55\,k + 55\,k^2\right)\left(5\,k^2 - 5\,k + 1\right)^2 (k-1)^3\,k^3$$

The above equation yields that $\delta(f)$ is zero only for $k \in \{0, \frac{1}{2} - \frac{3}{110} i\sqrt{55}, \frac{1}{2} + \frac{3}{110} i\sqrt{55}, \frac{1}{2} - \frac{1}{10}\sqrt{5}, \frac{1}{2} + \frac{1}{10}\sqrt{5}, 1\}$. However, for $k = 1$, and $k = 0$ the form $\Phi_f^k(x,y)$ only has the zero $(x,y) = (0,0)$ and the other possible values k for which f_k has a real zero are irrational. This in turn implies that for all natural numbers n there is no $k \in \{\frac{1}{n}, \ldots, \frac{n-1}{n}\}$ such that Φ^k has a real zero besides $(0,0)$. Therefore we can conclude that for any $n \in \mathbb{N}$ the form $f^{(n)}$ will be positive.

6. Symmetric mean inequalities of degree four

Now we turn to the question of characterizing quartic symmetric mean inequalities that are valid of all values of n. As it was defined in Section 2 the cone $\mathfrak{P}_4$ denotes the set of all sequences $\mathfrak{f} = (f^{(4)}, f^{(5)}, \ldots)$ of degree 4 means that are nonnegative for all n and $\mathfrak{S}_4$ the set of such sequences that can be written as sums of squares.

Let $\alpha \in [0,1]$. For a partition $\lambda := (\lambda_1, \ldots, \lambda_l) \vdash 4$ we define the bivariate form

$$\phi_\lambda^{(\alpha)} := \prod_{i=1}^{l} (\alpha x^{\lambda_i} + (1-\alpha) y^{\lambda_i}).$$

Lemma 6.1. *Let*

$$\mathfrak{f} := \sum_{\lambda \vdash 4} c_\lambda \mathfrak{p}_\lambda$$

be a linear combination of quartic symmetric power means. Then $\mathfrak{f} \in \mathfrak{P}_4$ if any only if for all $\alpha \in (0,1)$ the univariate polynomial

$$\Phi_{\mathfrak{f}}^{\alpha}(x,1) := \sum_{\lambda \vdash 4} c_\lambda \phi_\lambda^{(\alpha)}(x,1)$$

is nonnegative.

Proof. Suppose that $\mathfrak{f} \in \mathfrak{P}_4$. Then for all $n \geq 4$ the corresponding form $f^{(n)}$ in the sequence is nonnegative i.e., $f^{(n)}(x) \geq 0$ for all $x \in \mathbb{R}^n$. From Corollary 3.2 we know that $f^{(n)}(x) \geq 0$ if and only if the univariate polynomial $\Phi_{\mathfrak{f}}^{\alpha}(x,1)$ is nonnegative for all $\alpha \in \{\frac{1}{n} \ldots, \frac{n}{n}\}$. Hence the condition $f^{(n)} \geq 0$ for all $n \geq 4$ is equivalent to $\Phi_{\mathfrak{f}}^{\alpha}(x,1) \geq 0$ for all $\alpha \in \mathbb{Q} \cap [0,1]$.

Therefore $\Phi_{\mathfrak{f}}^{\alpha}(x,1)$ being nonnegative for all $\alpha \in [0,1]$ clearly implies that $\mathfrak{f} \in \mathfrak{P}_4$. On the other hand, suppose there is $\alpha_0 \in [0,1]$, $\alpha_0 \notin \mathbb{Q}$, with $\Phi_{\mathfrak{f}}^{\alpha_0}(t,1) < 0$ for some $t \in \mathbb{R}$. The roots of this univariate polynomial depend continuously on its coefficients. Therefore there exists $\varepsilon > 0$ such that for all $\alpha \in [0,1]$ with $|\alpha - \alpha_0| < \varepsilon$ we still have $\Phi_{\mathfrak{f}}^{\alpha_0}(t,1) < 0$. But since $\mathbb{Q}$ is dense in $[0,1]$ we see that there is $\alpha_1 \in \mathbb{Q} \cap [0,1]$ with $\Phi_{\mathfrak{f}}^{\alpha_0}(t,1) < 0$ which is a contradiction. □

Now we turn to the characterization of the elements on the boundary of $\mathfrak{P}_4$.

Lemma 6.2. *Let $0 \neq \mathfrak{f} \in \mathfrak{P}_4$. Then $\mathfrak{f}$ is on the boundary $\partial\mathfrak{P}_4$ if and only if there is $\alpha \in (0,1)$ such that the univariate polynomial $\Phi_{\mathfrak{f}}^{\alpha}(x,1)$ has a real double root.*

Proof. Let $\mathfrak{f} \in \partial\mathfrak{P}_4$. Suppose that for all $\alpha \in [0,1]$ the univariate polynomial $\Phi_{\mathfrak{f}}^{\alpha}(x,1)$ has only single roots. Since by Lemma 6.1 we have that $\Phi_{\mathfrak{f}}^{\alpha}(x,1)$ is a nonnegative polynomial for all $\alpha \in [0,1]$ all this root will have a non zero imaginary part. Hence it now follows that $\Phi_{\mathfrak{f}}^{\alpha}(x,1)$ is a strictly positive polynomial for all $\alpha \in [0,1]$. This in turn implies that for any sufficiently small perturbation $\tilde{\mathfrak{f}}$ of the coefficients c_λ the roots of $\Phi_{\tilde{\mathfrak{f}}}^{\alpha}(x,1)$ will still have non zero imaginary parts for all $\alpha \in [0,1]$ and thus any such $\Phi_{\tilde{\mathfrak{f}}}^{\alpha}(x,1)$ will be strictly positive. Hence by Lemma 6.1 a neighborhood of $\mathfrak{f}$ is in $\mathfrak{P}_4$, which clearly contradicts the assumption that $\mathfrak{f} \in \partial\mathfrak{P}_4$. Therefore there exists $\alpha \in [0,1]$ such that $\Phi_{\mathfrak{f}}^{\alpha}(x,1)$ has a real double root. Since for $\alpha = 0$ or $\alpha = 1$ this is only possible if $\mathfrak{f} = 0$, we have shown that $\alpha \in (0,1)$.

The other direction also follows from a continuity argument: Indeed, for any $\mathfrak{f}$ such that $\Phi_{\mathfrak{f}}^{\alpha}(x,1)$ has at least one real double root for some $\alpha \in (0,1)$ we can find by continuity of the roots for every $\varepsilon > 0$ a $\tilde{\mathfrak{f}} \notin \mathfrak{P}_4$ with $|\mathfrak{f} - \tilde{\mathfrak{f}}| < \varepsilon$ and hence we clearly have $\mathfrak{f} \in \partial\mathfrak{P}_4$. □

In order to algebraically characterize the elements on the boundary recall that the discriminant of a polynomial is a homogeneous polynomial in the coefficients, which vanishes exactly on the set of polynomials with multiple roots. So if $f(t) := \prod_i (x_i - t)$, we have $\operatorname{disc}(f) = \prod (x_i - x_j)$. However, note that $\operatorname{disc}(f) = 0$ alone does not guarantee that f has a real double root.

Theorem 6.3. *A symmetric inequality of degree 4 holds if and only if it can be written as a sum of squares for all n, i.e., we have*

$$\mathfrak{S}_4 = \mathfrak{P}_4.$$

Proof. Since $\mathfrak{S}_4 \subset \mathfrak{P}_4$ and both sets are closed convex cones, it suffices to show that there is no supporting hyperplane of $\mathfrak{S}_4$ which separates one element of $\mathfrak{P}_4$ from $\mathfrak{S}_4$. From Theorem 5.7 we deduce that a necessary condition for existence of such a hyperplane that there is a sequence $\mathfrak{f} := (f^{(4)}, f^{(5)}, \ldots)$ with $f^{(n)}$ as in Theorem 5.7, i.e.,

$$\mathfrak{f} = a^2\mathfrak{p}_4 + 2ab\mathfrak{p}_{31} + (c^2 - a^2)\mathfrak{p}_{22} + (2cd + b^2 - 2ab)\mathfrak{p}_{211} + (d^2 - b^2)\mathfrak{p}_{1111},$$

such that the coefficients a, b, c, d meet the conditions in (5.4), and such that $\mathfrak{f}$ is strictly in the interior of $\mathfrak{P}_4$. We now show that this is not possible, i.e. every $\mathfrak{f} \in \mathfrak{M}_4$ of the above form is on the boundary of $\mathfrak{P}_4$. We argue with Lemma 6.2. This amounts to verify that for any such $\mathfrak{f}$ there is an an α with $0 < \alpha < 1$ such $\delta_{\mathfrak{f}}(\alpha) = 0$ and that indeed the double root is real.

Using the formulae for univariate discriminants we calculate that this discriminants factors as

$$\delta_{\mathfrak{f}}(\alpha) = 16(\alpha - 1)^3(c + d)^2\alpha^3 Q_1(\alpha)Q_2(\alpha)^2,$$

where

$$Q_1 := \Gamma_1\alpha^2 - \Gamma_1\alpha - 16a^2(c + d)^2,$$

with

$$\Gamma_1 = \left(4\,cd - 8\,ad + 4\,a^2 + 4\,ab + b^2 - 8\,ca + 4\,c^2\right)\left(4\,cd + 8\,ad + 4\,a^2 + 4\,ab + b^2 + 8\,ca + 4\,c^2\right).$$

and

$$Q_2 := \Gamma_2\alpha^2 - \Gamma_2\alpha + a^2(c + d),$$

with $\Gamma_2 = 4a^2d - b^2c - 4abc$. We turn to a detailed examination of the factors and start with the following

Claim: Under the preconditions on a,b,c,d the polynomial Q_1 has no root for $\alpha\in(0,1)$ and Q_2 has always a root for $\alpha\in(0,1)$.

To proof the first part of the claim just observe that we have

$$Q_1(0)=Q_1(1)=-16\,a^2\,(c+d)^2<0.$$

Further we have

$$Q_1(\frac{1}{2})=-\frac{1}{4}\,\left(4\,a^2+4\,ab+4\,cd+4\,c^2+b^2\right)^2<0.$$

Which clearly implies $Q_1(\alpha)<0$ for all $\alpha\in(0,1)$.
To see the second claim we observe that

$$Q_2(0)=Q_2(1)=a^2(c+d)>0,$$

under the conditions on c,d. On the other hand we have

$$Q_2(\frac{1}{2})=\frac{1}{4}\,c\,(2\,a+b)^2<0$$

since c is supposed to be negative. This clearly implies that $Q_2(\alpha)$ has two real roots $\alpha_{1/2}\in(0,1)$. Therefore the polynomial $\Phi_{\mathfrak{f}}^{\alpha_{1/2}}(x,1)$ has a double root. However, it remains to verify that this double root is indeed real. In order to establish this we examine the situation of possible roots of Q_2 a little bite more carefully: Since Q_2 is a square, we have that $Q_2(a,b,c,d,\alpha)=0$ will also imply that the gradient of Q_2 is equal to zero. Setting

$$g_1:=a^2d+a^2c+4\,\alpha^2a^2d-4\,a^2\alpha d-4\,ab\alpha^2c+4\,ab\alpha c-\alpha^2b^2c+b^2\alpha c,$$

we obtain

$$grad(Q_2):=2g_1\cdot\begin{pmatrix}2\,ad+2\,ac+8\,\alpha^2ad-8\,a\alpha d-4\,b\alpha^2c+4\,b\alpha c\\ -4\,a\alpha^2c+4\,a\alpha c-2\,b\alpha^2c+2\,b\alpha c\\ a^2-4\,ab\alpha^2+4\,ab\alpha-\alpha^2b^2+b^2\alpha\\ a^2+4\,\alpha^2a^2-4\,a^2\alpha\\ a^2\alpha d-4\,a^2d-8\,ab\alpha c+4\,abc-2\,b^2\alpha c+b^2c\end{pmatrix}.$$

We conclude that any tuple (a,b,c,d,α) which is compatible with the conditions in (5.4) and for which $grad(Q_2)(a,b,c,d,\alpha)=0$ holds will thus either satisfy $g_1(a,b,c,d,\alpha)=0$ or will have $\alpha=\frac{1}{2}$ and $a=-\frac{1}{2}$.

In the case where $g_1(a,b,c,d,\alpha)=0$ we can deduce that $d=-\frac{c\left(a^2-4\,ab\alpha^2+4\,ab\alpha-\alpha^2b^2+b^2\alpha\right)}{a^2(2\,\alpha-1)^2}$. This yields that $\Phi_{\mathfrak{f}}^{\alpha^*}(x,1)$ contains the factor $(ax+a+b\alpha x-b\alpha+b)^2$. Thus under this condition we have a real double root at

$$x=-\frac{a+b-b\alpha}{a+b\alpha}.$$

In the second case we first observe that under the condition $a=-\frac{1}{2}$ and $\alpha=\frac{1}{2}$ the polynomial Q_2 specializes to $1/4\,c\,(-1+b)^2$. Thus the discriminant is zero if and only if $b=1$. Therefore we get under this conditions

$$\Phi_{\mathfrak{f}}^{\frac{1}{2}}(x,1)=\frac{1}{16}\,\left(d+2\,dx+2\,c+2\,cx^2+dx^2\right)^2$$

and we can establish that $x_{1/2}:=\pm\frac{-d+2\,\sqrt{-c(c-d)}}{2\,c+d}$ are the two double roots in this case. Since under the general conditions on c,d we have that $-c(c-d)\geq 0$ these will also be real. Therefore we have shown that all possible roots are indeed real. Since this shows that $\mathfrak{f} \in \partial\mathfrak{P}_4$ we can conclude, that $\mathfrak{S}_4 = \mathfrak{P}_4$.

□

In view of Corollary 4.6 it seems natural to ask if the situation of symmetric quartiles is somehow special or if Theorem 6.3 can be generalized to all degrees:

Question 6.4. *Let $d \in \mathbb{N}$. Is it true that we have $\mathfrak{P}_{2d} = \mathfrak{S}_{2d}$?*

Question 6.5. *In degree four we have that $\Sigma'_{n,4} = \mathfrak{S}_4$. How about a description of $\mathfrak{S}_{2d}$ in general?*

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