IN HOW MANY WAYS IS A QUARTIC A CONIC OF CONICS? DATA FOR IRREDUCIBLE QUARTICS OVER $\mathbb{C}$ AND FOR NONNEGATIVE ABSOLUTELY IRREDUCIBLE QUARTICS OVER $\mathbb{R}$

This collection of data is a supplement to the paper

[1] C. Scheiderer: Hilbert's theorem on positive ternary quartics: A refined analysis.
J. Algebraic Geometry **19**, 285–333 (2010)

Full details are given here to the situations analyzed in [1]. The presentation is divided in two sections, the "complex case" and the "real psd case". They can be considered as amplifications of [1] 4.4, Table 2, and of [1] 7.3, Table 4, respectively. Each section has a brief summary and explanation of notation. For detailed notation and setup we refer to [1].

1. The complex case

Let $0 \neq f \in \mathbb{C}[x_0, x_1, x_2]$ be irreducible and homogeneous of degree 4, let X be the plane curve defined by $f = 0$. Depending on the singularities of X, the table below gives, for each fixed base locus B, the number of equivalence classes of representations

$$f \;=\; p_1 p_2 - p_0^2 \qquad (det)$$

with the given base locus, in which the p_ν ($\nu = 0, 1, 2$) are quadratic forms in $\mathbb{C}[x_0, x_1, x_2]$. Recall that the base locus of (*det*) is the common zero scheme of p_0, p_1, p_2. This is a closed subscheme of X contained in the singular locus of X.

The table has seven columns which have the following meaning:

(1) $\mathrm{Sing}(X)$: The singularities of X. For reference in column (2), the singular points are labelled with letters P, Q etc. Different types of singularities get different letters, in the order in which they are listed. Singularities of the same type get the same letter, indexed by consecutive numbers. For example, when X has singularity type $A_1 + 2A_2$, the three singularities are labelled P, Q_1, Q_2, where P is a node and Q_1, Q_2 are cusps.

(2) Base locus B. We use infinitely near points to denote nonreduced base loci. If P is a (singular) point on X, then infinitely near points of the first order over P are denoted P' (or P'_1, $P'_2, \ldots$ in case there is more than one). Infinitely near points of the second order are written P'', etc. Given $\mathrm{Sing}(X)$, all base loci B are listed whose ideal sheaf arises as the conductor sheaf of some Gorenstein partial normalization $X' \to X$ (uniquely determined by B, see [1] Sect. 2, in particular 2.14, 2.22). For the list of these B see *loc. cit.*, 3.10. In the sequel, $\pi\colon X' \to X$ denotes the partial normalization associated to B.

(3) $\mathrm{Sing}(X')$: The singularities of X'.

(4) $|J'_2|$: The order of the 2-torsion subgroup of the generalized Jacobian J' of X'.

(5) irrelev.: The number of irrelevant line bundles $\mathcal{F}'$ on X', that is, for which either $\mathcal{F}' = \mathcal{O}_X(1)$ (if $X' = X$, i. e., $B = \varnothing$), or $\mathcal{F}' = \pi^o\mathcal{F}$ where $\mathcal{F}$ is an exceptional sheaf on X (see [1] 3.3, 3.6, 3.11).

(6) det. rep. with B: The number of representations (*det*) with base locus B; this is the difference (4) – (5) (see [1] 4.4).

(7) det. rep. total: The total number of representations (*det*) of f.

Sing(X)	Base locus B	Sing(X')	$\|J_2'\|$	irrelev.	det. rep. with B	det. rep. total
smooth	$\varnothing$	smooth	64	1	63	63
A_1	$\varnothing$	A_1	32	2	30	46
	P	smooth	16		16	
A_2	$\varnothing$	A_2	16	1	15	30
	P	smooth	16	1	15	
A_3	$\varnothing$	A_3	8	1	7	18
	P	A_1	8		8	
	P, P'	smooth	4	1	3	
A_4	$\varnothing$	A_4	4	1	3	10
	P	A_2	4		4	
	P, P'	smooth	4	1	3	
A_5	$\varnothing$	A_5	2	1	1	5
	P	A_3	2		2	
	P, P'	A_1	2	1	1	
	P, P', P''	smooth	1		1	
A_6	$\varnothing$	A_6	1	1	—	2
	P	A_4	1		1	
	P, P'	A_2	1	1	—	
	P, P', P''	smooth	1		1	
D_4	$\varnothing$	D_4	4	1	3	9
	P, P_i' $(3\times)$	A_1	2		2 $(3\times)$	
	P, P_1', P_2', P_3'	smooth	1	1	—	
D_5	$\varnothing$	D_5	2	1	1	4
	P, P_1'	A_1	2		2	
	P, P_2'	A_2	1		1	
	P, P_1', P_2'	smooth	1	1	—	
E_6	$\varnothing$	E_6	1	1	—	1
	P, P'	A_2	1		1	
	P, P', P''	smooth	1	1	—	
$2A_1$	$\varnothing$	$2A_1$	16	3	13	33
	P_i $(2\times)$	A_1	8		8 $(2\times)$	
	P_1, P_2	smooth	4		4	
$A_1 + A_2$	$\varnothing$	$A_1 + A_2$	8	2	6	21
	P	A_2	4		4	
	Q	A_1	8	1	7	
	P, Q	smooth	4		4	
$A_1 + A_3$	$\varnothing$	$A_1 + A_3$	4	2	2	12
	P	A_3	2		2	
	Q	$2A_1$	4		4	
	Q, Q'	A_1	2	1	1	
	P, Q	A_1	2		2	
	P, Q, Q'	smooth	1		1	

Sing(X)	Base locus B	Sing(X')	$\lvert J_2'\rvert$	irrelev.	det. rep. with B	det. rep. total
A_1+A_4	$\varnothing$	A_1+A_4	2	2	—	6
	P	A_4	1		1	
	Q	A_1+A_2	2		2	
	Q,Q'	A_1	2	1	1	
	P,Q	A_2	1		1	
	P,Q,Q'	smooth	1		1	
$2A_2$	$\varnothing$	$2A_2$	4	1	3	13
	P_i (2×)	A_2	4	1 (2×)	3 (2×)	
	P_1,P_2	smooth	4		4	
A_2+A_3	$\varnothing$	A_2+A_3	2	1	1	7
	P	A_3	2	1	1	
	Q	A_1+A_2	2		2	
	Q,Q'	A_2	1	1	—	
	P,Q	A_1	2		2	
	P,Q,Q'	smooth	1		1	
A_2+A_4	$\varnothing$	A_2+A_4	1	1	—	3
	P	A_4	1	1	—	
	Q	$2A_2$	1		1	
	Q,Q'	A_2	1	1	—	
	P,Q	A_2	1		1	
	P,Q,Q'	smooth	1		1	
$3A_1$	$\varnothing$	$3A_1$	8	4	4	23
	P_i (3×)	$2A_1$	4		4 (3×)	
	P_i,P_j (3×)	A_1	2		2 (3×)	
	P_1,P_2,P_3	smooth	1		1	
$2A_1+A_2$	$\varnothing$	$2A_1+A_2$	4	3	1	14
	P_i (2×)	A_1+A_2	2		2 (2×)	
	Q	$2A_1$	4	1	3	
	P_i,Q (2×)	A_1	2		2 (2×)	
	P_1,P_2	A_2	1		1	
	P_1,P_2,Q	smooth	1		1	
A_1+2A_2	$\varnothing$	A_1+2A_2	2	2	—	8
	P	$2A_2$	1		1	
	Q_i (2×)	A_1+A_2	2	1 (2×)	1 (2×)	
	P,Q_i (2×)	A_2	1		1 (2×)	
	Q_1,Q_2	A_1	2		2	
	P,Q_1,Q_2	smooth	1		1	
$3A_2$	$\varnothing$	$3A_2$	1	1	—	4
	P_i (3×)	$2A_2$	1	1 (3×)	— (3×)	
	P_i,P_j (3×)	A_2	1		1 (3×)	
	P_1,P_2,P_3	smooth	1		1	

2. The real psd case

Let $0 \neq f \in \mathbb{R}[x_0, x_1, x_2]$ be homogeneous of degree 4, positive semidefinite (psd) and irreducible over $\mathbb{C}$, and let X be the plane curve over $\mathbb{R}$ defined by $f = 0$. Depending on the singularities of X, the table below gives, for each fixed base locus B, the number of equivalence classes of signed representations

$$f = \pm p_0^2 \pm p_1^2 \pm p_2^2 \tag{sig}$$

and of sums of squares (sos) representations

$$f = p_0^2 + p_1^2 + p_2^2 \tag{sos}$$

with the given base locus, where the $p_\nu \in \mathbb{R}[x_0, x_1, x_2]$ ($\nu = 0, 1, 2$) are quadratic forms.

The table has twelve columns which have the following meaning:

(1) Sing(X): The singularities of X. The same conventions apply for their labelling as in the complex case (see above). The notation for the real singularity types follows the convention in [1] 6.7.
(2) Base locus B: See the complex case (above) for notational conventions. In the sequel, let $\pi\colon X' \to X$ denote the partial normalization of X corresponding to B.
(3) Sing(X'): The singularities of X'.
(4) $|J'(\mathbb{R})_2|$: The order of the 2-torsion subgroup of $J'(\mathbb{R})$, where J' is the generalized Jacobian of X' (see [1] 6.14).
(5) ∂'_2 denotes the restriction $J'(\mathbb{R})_2 \to \mathrm{Br}(\mathbb{R})$ of the map $\partial\colon \mathrm{Pic}(X'_{\mathbb{C}})^G \to \mathrm{Br}(\mathbb{R})$ ([1] 5.15). It is listed here whether ∂'_2 is zero ("0") or surjective ("sur") (see [1] 6.10).
(6) $D : G$: It is listed here whether or not $\pi^!\mathcal{O}_X(2)$ is a double in $\mathrm{Pic}(X'_{\mathbb{C}})^G$ ("y" for yes, "n" for no) (see [1] 7.1).
(7) D: Similarly, it is listed here whether $\pi^!\mathcal{O}_X(2)$ is a double in $\mathrm{Pic}(X')$ ("y" for yes, "n" for no) (see [1] 7.1).
(8) irrelev.: The number of irrelevant line bundles $\mathcal{F}'$ on X' (see the complex case above).
(9) signed with B: The number of signed representations (sig) with base locus B (see [1] 7.2).
(10) sos with B: The number of sos representations (sos) with base locus B (see [1] 7.2).
(11) signed total: The total number of signed representations (sig) of f.
(12) sos total: The total number of sos representations (sos) of f.

Sing(X)	Base locus B	Sing(X')	$\lvert J'(\mathbb{R})_2\rvert$	∂_2'	$D{:}G$	D	irrelev.	signed with B	sos with B	signed total	sos total
smooth	$\varnothing$	smooth	16	sur	y	y	1	15	8	15	8
A_1^*	$\varnothing$	A_1^*	8	0	y	y	2	6	—	10	4
	P	smooth	4	0	y	n		4	4		
$2A_1^*$	$\varnothing$	$2A_1^*$	8	0	y	y	3	5	—	9	2
	P_i (2×)	A_1^*	4	0	n	n		—	—		
	P_1, P_2	smooth	4	sur	y	y		4	2		
$2A_1^i$	$\varnothing$	$2A_1^i$	8	sur	y	y	1	7	4	11	6
	$P, \overline{P}$	smooth	4	sur	y	y		4	2		
$3A_1^*$	$\varnothing$	$3A_1^*$	8	0	y	y	4	4	—	11	1
	P_i (3×)	$2A_1^*$	4	0	n	n		—	—		
	P_i, P_j (3×)	A_1^*	2	0	y	y		2 (3×)	—		
	P_1, P_2, P_3	smooth	1	0	y	n		1	1		
$A_1^* + 2A_1^i$	$\varnothing$	$A_1^* + 2A_1^i$	4	0	y	y	2	2	—	7	3
	P	$2A_1^i$	2	0	y	n		2	2		
	$Q, \overline{Q}$	A_1^*	2	0	y	y		2	—		
	$P, Q, \overline{Q}$	smooth	1	0	y	n		1	1		
$2A_2^i$	$\varnothing$	$2A_2^i$	4	sur	y	y	1	3	2	7	4
	$P, \overline{P}$	smooth	4	sur	y	y		4	2		
$A_1^* + 2A_2^i$	$\varnothing$	$A_1^* + 2A_2^i$	2	0	y	y	2	—	—	4	2
	P	$2A_2^i$	1	0	y	n		1	1		
	$Q, \overline{Q}$	A_1^*	2	0	y	y		2	—		
	$P, Q, \overline{Q}$	smooth	1	0	y	n		1	1		
A_3^*	$\varnothing$	A_3^*	4	0	y	y	1	3	—	6	2
	P	A_1^*	4	0	n	n		—	—		
	P, P'	smooth	4	sur	y	y	1	3	2		
$A_1^* + A_3^*$	$\varnothing$	$A_1^* + A_3^*$	4	0	y	y	2	2	—	6	1
	P	A_3^*	2	0	n	n		—	—		
	Q	$2A_1^*$	4	0	n	n		—	—		
	P, Q	A_1^*	2	0	y	y		2	—		
	Q, Q'	A_1^*	2	0	y	y	1	1	—		
	P, Q, Q'	smooth	1	0	y	n		1	1		
A_5^*	$\varnothing$	A_5^*	2	0	y	y	1	1	—	3	1
	P	A_3^*	2	0	n	n		—	—		
	P, P'	A_1^*	2	0	y	y	1	1	—		
	P, P', P''	smooth	1	0	y	n		1	1		