A SHORT REMARK ON A THEOREM BY BECKER AND GONDARD

CLAUS SCHEIDERER

Let X be a smooth projective irreducible $\mathbb{R}$-variety, let $K = \mathbb{R}(X)$ be its function field, and let s denote the number of connected components of the space $X(\mathbb{R})$ of real points on X. This short remark provides a direct geometric proof for the following theorem:

Theorem (E. Becker and D. Gondard)**.** *Assume that $X(\mathbb{R})$ is non-empty. Then the quotient of the group $K^{*2} \cap \Sigma K^{*4}$ by its subgroup $(\Sigma K^{*2})^2$ is finite, of order 2^{s-1}.*

The result is obtained from Becker-Gondard's paper [3] by combining Corollary 4.4 with Theorem 3.8. Corollary 4.4 says, for any real field K, that the number of connected components of $M(K)$ is

$$1 + \log_2 \bigl[K^{*2} \cap \Sigma K^{*4} : (\Sigma K^{*2})^2\bigr],$$

where $M(K)$ is the compact topological space of real (i.e., $\mathbb{R}$-valued) places of K. If $K = \mathbb{R}(X)$ is the function field of a smooth projective irreducible $\mathbb{R}$-variety X, then the connected components of $M(K)$ are in canonical bijection with the connected components of $X(\mathbb{R})$, as shown in [3] Theorem 3.8.

The proof of [3] Corollary 4.4 makes essential use of Becker's theory of the real holomorphy ring, as exposed in [2]. Both that proof and the one given below use Becker's valuation-theoretic characterization of sums of higher powers [1] to show that $(\Sigma K^{*2})^2$ is contained in ΣK^{*4}.

Alternative proof of the theorem. Let $f \in K^*$ be such that f^2 is a sum of fourth powers. Then f does not change sign on any connected component W of $X(\mathbb{R})$. In other words, if P_1, P_2 are any two points in W where f is defined, then $f(P_1)f(P_2) \geq 0$. Otherwise, f would change sign along a real irreducible subvariety Y of X of codimension one, and hence $v_Y(f)$ would be odd. (Here Y real means that $Y(\mathbb{R})$ is Zariski dense in Y, or equivalently, that $\mathbb{R}(Y)$ is a real field; and v_Y denotes the discrete valuation of K associated with Y.) It would follow that $v_Y(f^2) \equiv 2 \pmod 4$, and hence f^2 could not be a sum of fourth powers (this uses that Y is real).

Let S be the set of connected components of $X(\mathbb{R})$. The preceding argument shows that we get a well-defined group homomorphism

$$K^{*2} \cap \Sigma K^{*4} \to \{\pm 1\}^S / \pm 1, \qquad f^2 \mapsto \pm\sigma(f), \tag{$*$}$$

where $\sigma(f) \in \{\pm 1\}^S$ is the sign distribution on S given by the locally semidefinite map f. The kernel of $(*)$ consists of the squares of non-negative rational functions, and so it is $(\Sigma K^{*2})^2$ by the theorem of Artin and Lang. The map $(*)$ is also surjective. To see this, let $X(\mathbb{R}) = W_1 \cup W_2$ be a disjoint decomposition into open and closed sets W_1, W_2. There is a rational function $f \in K^*$ which is defined everywhere on $X(\mathbb{R})$ such that $f|_{W_1} > 0$ and $f|_{W_2} < 0$. (For example, choose an affine open subset X_0 of X with $X_0(\mathbb{R}) = X(\mathbb{R})$ and argue by Weierstraß approximation.) By Becker's valuation-theoretic characterization of sums of even powers [1], f^2 is a sum of $2n$-th powers for any $n \geq 1$. It is clear that f^2 maps under $(*)$ to the sign distribution given by W_1 and W_2. $\square$

References

[1] E. Becker: Summen n-ter Potenzen in Körpern. J. reine angew. Math. **307/308**, 8–30 (1979).

[2] E. Becker: The real holomorphy ring and sums of $2n$-th powers. In: *Géométrie Algébrique Réelle et Formes Quadratiques* (Rennes, 1981), ed. J.-L. Colliot-Thélène, M. Coste, L. Mahé, M.-F. Roy, Lect. Notes Math. **959**, Springer, Berlin 1982, pp. 139–181.

[3] E. Becker, D. Gondard: Notes on the space of real places of a formally real field. In: *Real analytic and algebraic geometry* (Trento, 1992), ed. F. Broglia, M. Galbiati, A. Tognoli, de Gruyter, Berlin 1995, pp. 21–46.

Berkeley, 15 April 2004

Institut für Mathematik, Fakultät 4, Universität Duisburg, 47048 Duisburg, Germany

E-mail address: `claus@uni-duisburg.de`