CORRECTION TO THEOREM 6.4 IN "SUMS OF SQUARES OF REGULAR FUNCTIONS..."

This note refers to the paper

[0] C. Scheiderer: Sums of squares of regular functions on real algebraic varieties. Trans. Am. Math. Soc. **352**, 1039–1069 (1999).

Theorem 6.4 from [0] says the following:

Theorem. *Let V be a smooth affine surface over R which has a smooth compactification $\overline{V}$ for which every irreducible curve contained in $\overline{V} \smallsetminus V$ is real. Then the preorder of all psd elements in $R[V]$ is not finitely generated. In particular, $R[V]$ contains psd elements which are not sums of squares.*

The theorem is correct. However, its proof, as given in [0], has some deficencies in the part which proves Lemma 6.5. The argument in the end is not explained very clearly. More serious, however, is the fact that the proof of Lemma 6.5 is not correct in this generality. The proof can be saved if one adds suitable conditions on the nature of the singularities of the curves C_i. But, as stated, Lemma 6.5 might even be false, since the (linear equivalence class of the) very ample divisor D_0 is prescribed. The problem is that, given a real projective integral curve C which is singular, there may exist effective divisors D such that no multiple nD, $n \geq 1$, is linearly equivalent to an effective, totally real divisor whose support consists of regular points.

The aim of this note is to rectify this problem. To do so, we first desingularize the curves in $\overline{V} - V$. In this way we can reduce to a situation in which the proof of Lemma 6.5 presents no problem.

The proof of Theorem 6.4 will follow from the following

Proposition. *Let X be a nonsingular proper surface over R, and let Y be a curve on X whose irreducible components are all real. Moreover, let a finite subset S of $Y(R)$ be given. Then there exists a nonsingular irreducible curve C on X of genus > 0 whose points of intersection with Y are all real and lie outside of S.*

Assuming the proposition, we deduce the theorem as in [0]: Let $Y := \overline{V} - V$ (with the reduced subscheme structure). This is a curve (i.e., each irreducible component of Y is one-dimensional) since V is affine (see for example [14a] II.3.1 p. 66.[1]). By the proposition, we find a nonsingular curve C of positive genus on $\overline{V}$ which intersects Y in real points. Put $C' := C \cap V$. Let finitely many psd functions $f_1, \dots, f_n$ in $R[V]$ be given. By Theorem 3.4, since every point of C' at infinity is real, we can find a psd function $g \in R[C']$ which is not contained in the preorder generated by $f_1|_{C'}, \dots, f_n|_{C'}$ in $R[C']$. By Theorem 5.6 we can extend g to a psd function f in $R[V]$. This f cannot lie in the preorder generated by $f_1, \dots, f_n$ in $R[V]$.

Proof of the proposition. Other than in [0], we first regularize the situation. By making iterated blowing-ups of X in points which are infinitely near to points in Y_{sing}, we arrive at a birational morphism $\pi\colon X' \to X$ of proper R-varieties which is an isomorphism over $X - Y_{\text{sing}}$, in such a way that the strict transform Y' of Y in

Date: 24 January, 2004.

[1] In addition it is known that Y is connected (loc.cit. II.6.2 p. 79).

X' is nonsingular (and hence is the normalization of the curve Y). Now it suffices to find an irreducible nonsingular curve C' in X' of genus > 0 whose intersection with Y' consists of real points which do not lie over points from $S \cup Y_{\text{sing}}$. Indeed, setting $C := \pi(C')$, the restriction $C' \to C$ of π is an isomorphism, and C intersects the curve Y in real points only.

In other words, for the proof of the proposition we can assume that the curve Y on X is nonsingular (not necessarily connected).

To proceed further we need the following lemma. It is a corrected version of Lemma 6.5 from [0]. (The lemma is true in situations more general than considered here, but is probably false in the generality which was claimed in [0].)

Lemma. *Let X be a proper nonsingular surface over R, and let Y be a nonsingular curve on X whose irreducible components are all real. Let a finite subset S of $Y(R)$ be given, and let D_0 be a very ample effective divisor on X. Then there is an integer $N \geq 1$ with the following property: For every $n > N$ there is a divisor $D \in |nD_0|$ which meets Y transversely in real points outside of S.*

Proof of the proposition (continued). We first finish the proof of the proposition. Since X is projective, the lemma gives us a very ample effective divisor D on X which meets Y transversely in real points outside of S. The elements of the complete linear system $|D|$ with this property form an open subset of $|D|$ (with respect to the canonical semi-algebraic topology on $|D|$). A Bertini argument (see [14] ch. V, Lemma 1.2) shows that we can even find a D with this property which is irreducible and nonsingular. By replacing D with mD for suitable $m \geq 1$ (and again applying the lemma and the Bertini argument), we can at the same time make the genus g_D arbitrarily big (and, in particular, strictly positive), according to the adjunction formula $2g_D - 2 = D \cdot (D + K)$, and since $D^2 > 0$. □

Proof of the lemma. Let $Y_1, \ldots, Y_r$ be the irreducible components of Y. After replacing D by a linearly equivalent effective divisor, we can assume that none of the Y_i is contained in $\text{supp}(D)$, and that $D \cap S = \varnothing$. The intersection $D \cap Y$ is an effective divisor on Y, und for each $i = 1, \ldots, r$ we have $\deg(D \cap Y_i) = D \cdot Y_i > 0$. Applying Corollary 2.10 and Remark 2.14 to each of $Y_1, \ldots, Y_r$, we conclude: There is $N_1 \geq 1$ such that for every $n > N_1$ and every $i = 1, \ldots, r$ there is a divisor $E_i \in |nD \cap Y_i|$ (linear system on Y_i) which has the form $[P_1] + \cdots + [P_k]$ with pairwise different real points $P_1, \ldots, P_k \in Y_i(R)$, $P_\nu \notin S$.

On the other hand, let $\mathcal{L} = \mathcal{O}_X(D)$ be the invertible sheaf on X defined by D. Since $\mathcal{L}$ is ample, we have $H^1(X, \mathfrak{I}_Y \otimes \mathcal{L}^n) = 0$ für $n \gg 0$. Hence there is $N_2 \geq 1$ such that $H^0(X, \mathcal{L}^{\otimes n}) \to H^0(Y, \mathcal{L}|_Y^{\otimes n})$ is surjective for $n > N_2$. It follows that if $n > N_2$, then for any given tuple $(E_1, \ldots, E_r)$ of divisors $E_i \in |nD \cap Y_i|$ $(i = 1, \ldots, r)$ there is a divisor $\Theta \in |nD|$, not containing any of the Y_i, with $E_i = \Theta \cap Y_i$ for $i = 1, \ldots, r$.

Altogether we conclude that for any $n > \max\{N_1, N_2\}$ there exists a divisor $\Theta \in |nD|$ which intersects Y transversely in real points which are not contained in S. This completes the proof of the lemma, and hence also of the theorem. □

References

[14] R. Hartshorne: *Algebraic Geometry.* Graduate Texts in Mathematics **52**, Springer, New York, 1977.

[14a] R. Hartshorne: *Ample Subvarieties of Algebraic Varieties.* Lecture Notes in Mathematics **156**, Springer, New York, 1970.