A REMARK ON DESCENDING SUMS OF SQUARES REPRESENTATIONS

CLAUS SCHEIDERER

Let $f = f(x_1, \dots, x_n)$ be a polynomial with rational coefficients which is a sum of squares (*sos*) of polynomials with real coefficients. Sturmfels asked whether f is necessarily a sum of squares of polynomials with rational coefficients. From the assumption it follows easily that f is a sum of squares of polynomials with real algebraic coefficients. Hence there exists a real number field K such that f is sos in $K[x_1, \dots, x_n]$.

In [1], Hillar showed that the answer to Sturmfels's question is positive if the real number field K can be chosen to be Galois over $\mathbb{Q}$. Moreover, he provided bounds for the number of squares needed to write f over $\mathbb{Q}$, in terms of the number of squares needed over K and the degree $[K : \mathbb{Q}]$. In fact, the polynomial ring $\mathbb{Q}[x_1, \dots, x_n]$ could be replaced for these results by any commutative $\mathbb{Q}$-algebra A (and accordingly $K[x_1, \dots, x_n]$ by $A \otimes_{\mathbb{Q}} K$).

The purpose of this note is to give a very short proof of a generalization of this result. This proof yields a significantly smaller bound for the number of squares necessary to express f as a sum of squares over $\mathbb{Q}$. While in [1] this bound is exponential in the degree $[K : \mathbb{Q}]$, our bound is linear.

We prefer to work over an arbitrary real base field k, since there is no difference in the proof.

Given a finite extension K/k of real fields, consider the associated trace quadratic form. This is the quadratic form

$$\tau \colon K \to k, \quad y \mapsto \mathrm{tr}_{K/k}(y^2)$$

over k. It has the following well-known basic property: For any ordering P of k, the Sylvester signature of τ at P is equal to the number of extensions of the ordering P to K. See [2] (Lemma 3.2.7 or Theorem 3.4.5), for example.

For any commutative ring A denote by ΣA^2 the set of sums of squares of A. Assume that every ordering of k has $d := [K : k]$ different extensions to K. (It is easy to see that this is equivalent to the condition that every ordering of k extends to the Galois hull of K/k.) Then τ is positive definite with respect to every ordering of k. Diagonalizing τ therefore gives $a_1, \dots, a_d \in \Sigma k^2$, together with a k-linear basis $y_1, \dots, y_d$ of K, such that

$$\mathrm{tr}_{K/k}\Big(\Big(\sum_{i=1}^{d} x_i y_i\Big)^2\Big) \;=\; \sum_{i=1}^{d} a_i x_i^2 \tag{1}$$

holds for all $x_1, \dots, x_d \in k$. More generally, if A is an arbitrary (commutative) k-Algebra and $A_K := A \otimes_k K$, then

$$\mathrm{tr}_{A_K/A}\Big(\Big(\sum_{i=1}^{d} x_i \otimes y_i\Big)^2\Big) \;=\; \sum_{i=1}^{d} a_i x_i^2 \tag{2}$$

holds for all $x_1, \dots, x_n \in A$.

Date: July 16, 2009.
1991 *Mathematics Subject Classification.* Primary secondary .

Proposition 1. *Let K/k be an extension of real fields of finite degree $d = [K : k]$, and assume that every ordering of k extends to d different orderings of K. Then there exist $c_1, \dots, c_d \in \Sigma k^2$ with the following property:*

For every k-algebra A and every $f \in A$ which is a sum of m squares in $A_K = A \otimes_k K$, there are $f_1, \dots, f_d \in A$ such that each f_i is a sum of m squares in A, and such that

$$f = \sum_{i=1}^{d} c_i f_i.$$

In particular, f is a sum of $dm \cdot p(k)$ squares in A.

Here $p(k)$ denotes the Pythagoras number of k, i. e., the smallest number p such that every sum of squares in k is a sum of p squares in k. (If no such number p exists one puts $p(k) = \infty$.)

Proof. Choose $a_i \in \Sigma k^2$ and $y_i \in K$ $(i = 1, \dots, d)$ as before. It suffices to take $c_i = \frac{a_i}{d}$ for $i = 1, \dots, d$. Indeed, assuming $f = g_1^2 + \cdots + g_m^2$ with $g_1, \dots, g_m \in A_K$, we get

$$df = \operatorname{tr}_{A_K/A}(f) = \sum_{j=1}^{m} \operatorname{tr}_{A_K/A}(g_j^2) = \sum_{j=1}^{m} \sum_{i=1}^{d} a_i x_{ij}^2,$$

where the $x_{ij} \in A$ are determined by $g_j = \sum_{i=1}^{d} x_{ij} \otimes y_i$ $(j = 1, \dots, m)$. So we can put $f_i = \sum_{j=1}^{m} x_{ij}^2$ $(i = 1, \dots, d)$. □

Remark. In [1] it was shown (for $k = \mathbb{Q}$) that if $K/\mathbb{Q}$ is a totally real number field with Galois hull $L/\mathbb{Q}$, if A is a $\mathbb{Q}$-algebra and $f \in A$ is a sum of m squares in $A \otimes_{\mathbb{Q}} K$, then f is a sum of

$$4m \cdot 2^{e+1} \binom{e+1}{2} \;=\; 2^{e+2} e(e+1) \cdot m$$

squares in A, with $e := [L : \mathbb{Q}]$.

The qualitative part of the above result extends immediately to the following more general situation. Let K/k be an extension as in the proposition, and let A be a k-algebra. Fix elements $h_1, \dots, h_r \in A$ and consider the so-called (pseudo-) quadratic module

$$M \;:=\; \Big\{ \sum_{i=1}^{r} s_i h_i \colon s_1, \dots, s_r \in \Sigma A^2 \Big\}$$

generated by the h_i. Similarly, let

$$M_K \;=\; \Big\{ \sum_{i=1}^{r} t_i h_i \colon t_1, \dots, t_r \in \Sigma A_K^2 \Big\}$$

be the (pseudo-) quadratic module generated by M in A_K. Then we have:

Proposition 2. *$A \cap M_K = M$.* □

References

[1] Ch. J. Hillar: Sums of squares over totally real fields are rational sums of squares. Proc. Am. Math. Soc. **137**, 921–930 (2009).

[2] W. Scharlau: *Quadratic and Hermitian Forms.* Grundl. math. Wiss. **270**, Springer, Berlin, 1985.

Fachbereich Mathematik und Statistik, Universität Konstanz, 78457 Konstanz, Germany

E-mail address: claus.scheiderer@uni-konstanz.de