Corrigé de la feuille 3.

1. (a) $\int (x^2+\frac{1}{x^2})^2\,dx = \int (x^4+2+\frac{1}{x^4})\,dx$

$= \frac{x^5}{5} + 2x + \frac{x^{-3}}{-3} = \frac{x^5}{5}+2x-\frac{1}{3x^3}$

(b) $\int (\sin x)^2\,dx = \int \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^2 dx$

$= -\int \left(\frac{(e^{ix})^2}{4} - \frac{1}{2}e^{ix}e^{-ix} + \frac{(e^{-ix})^2}{4}\right) dx$

$= -\int \left(\frac{e^{2ix}}{4} - \frac{1}{2} + \frac{e^{-2ix}}{4}\right) dx$

$= -\int \left(\frac{\cos(2x)}{2} - \frac{1}{2}\right) dx$

$= \frac{x}{2} - \frac{1}{2}\int \cos(2x)\,dx$

$t = 2x$, $\frac{dt}{dx} = 2$, $dx = \frac{dt}{2}$

$\frac{x}{2} - \frac{1}{4}\int \cos t\,dt$

$= \frac{x}{2} - \frac{1}{4}(\sin t)$

$= \frac{x}{2} - \frac{1}{4}\sin(2x)$

2. (a) $\int \underbrace{x^2}_{u'}\underbrace{(\ln x)}_{v}\, dx = \underbrace{\frac{x^3}{3}}_{u}\cdot\underbrace{\ln x}_{v} - \int \underbrace{\frac{x^3}{3}}_{u}\cdot\underbrace{\frac{1}{x}}_{v'}\, dx$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2\, dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{x^3}{9}$$

(b) $\int \underbrace{x}_{u}\underbrace{e^x}_{v'}\, dx = \underbrace{x}_{u}\cdot\underbrace{e^x}_{v} - \int \underbrace{1}_{u'}\cdot\underbrace{e^x}_{v}\, dx$

$$= xe^x - e^x = (x-1)e^x$$

(c) $\int \ln(x)\, dx = \int \underbrace{1}_{u'}\cdot\underbrace{\ln(x)}_{v}\, dx = \underbrace{x}_{u}\underbrace{\ln x}_{v} - \int \frac{x}{x}\, dx$

$$= x\ln x - \int 1\, dx = x\ln x - x$$

$$= x(\ln x - 1)$$

(d) $\int \underbrace{x}_{u}\underbrace{\sin(x)}_{v'}\, dx = \underbrace{x}_{u}\underbrace{(-\cos x)}_{v} - \int \underbrace{1}_{u'}\cdot\underbrace{(-\cos x)}_{v}\, dx$

$$= -x\cos x + \sin x$$

$$= \sin x - x\cos x$$

3. (a) $\int \frac{x^5}{1+x^6}\, dx \overset{t = x^6+1}{\underset{\frac{dt}{dx}=6x^5}{=}} \frac{1}{6}\int \frac{1}{t}\, dt = \frac{1}{6}\ln|t|$

$$x^5 dx = \frac{dt}{6}$$

$$= \frac{1}{6}\ln|1+x^6| = \frac{1}{6}\ln(1+x^6)$$

(b) $\int (\sin x)(\cos x)\, dx \overset{t=\sin x}{\underset{\frac{dt}{dx}=\cos x}{=}} \int t\, dt = \frac{t^2}{2} = \frac{(\sin x)^2}{2}$

$$dt = (\cos x)\, dx$$

4. (a) $\int \frac{x}{\sqrt{x+1}}\,dx \underset{\frac{dt}{dx}=\frac{1}{2}(x+1)^{-\frac{1}{2}}}{\overset{t=\sqrt{x+1}}{=}} \int \frac{t^2-1}{t}\,2t\,dt$

$$t^2 = x+1$$
$$x = t^2-1$$
$$dx = 2\sqrt{x+1}\,dt$$
$$= 2\,t\,dt$$

$$= 2\int (t^2-1)dt = 2\left(\frac{t^3}{3}-t\right) = 2\sqrt{x+1}\left(\frac{x+1}{3}-1\right)$$

$$= \frac{2}{3}\sqrt{x+1}\,(x-2)$$

(b) $\int \frac{1}{3x^2+2}\,dx \underset{\frac{dt}{dx}=\sqrt{\frac{3}{2}}}{\overset{t=\sqrt{\frac{3}{2}}x}{=}} \sqrt{\frac{2}{3}}\int \frac{1}{2t^2+2}\,dt$

$$dx = \sqrt{\frac{2}{3}}\,dt$$
$$x^2 = \frac{2}{3}t^2$$

$$= \frac{1}{\sqrt{6}}\int \frac{1}{1+t^2}\,dt = \frac{1}{\sqrt{6}}\arctan t$$

$$= \frac{\arctan\left(\sqrt{\frac{3}{2}}\,x\right)}{\sqrt{6}}$$

5. $\int x\exp(x^2)\,dx \underset{\frac{dt}{dx}=2x}{\overset{t=x^2}{=}} \int (\exp t)\,\frac{dt}{2} = \frac{1}{2}\exp t = \frac{\exp(x^2)}{2}$

$$x\,dx = \frac{dt}{2}$$

6. $\int (A\exp(-\vartheta t) - B\exp(-\lambda t))\,dt$

$$= A\,\frac{\exp(-\vartheta t)}{-\vartheta} - B\,\frac{\exp(-\lambda t)}{-\lambda}$$

$$= \frac{B}{\lambda}\exp(-\lambda t) - \frac{A}{\vartheta}\exp(-\vartheta t)$$

La quantité totale de substance au temps $t \geqslant 0$ est donc

$$\frac{B}{\lambda}\exp(-\lambda t) - \frac{A}{\vartheta}\exp(-\vartheta t) + C$$

où $C \in \mathbb{R}$ est une constante à déterminer.
On sait qu'au temps 0 il n'y a pas encore de substance dans le sang, c.-à-d.

$$\frac{B}{\lambda}\underbrace{\exp(-\lambda\cdot 0)}_{=1} - \frac{A}{\vartheta}\underbrace{\exp(-\vartheta\cdot 0)}_{=1} + C = 0$$

d'où $C = \frac{A}{\vartheta} - \frac{B}{\lambda}$. La quantité totale est donc

$$\frac{B}{\lambda}(\exp(-\lambda t) - 1) + \frac{A}{\lambda}(1 - \exp(-\vartheta t))$$

au temps $t \geq 0$.

7. $\int (200 + 50t)\,dt = 200t + 25t^2$

$$\int_4^6 (200+50t)\,dt = \left[200t + 25t^2\right]_{t=4}^6 = (200\cdot 6 + 25\cdot 6^2) - (200\cdot 4 + 25\cdot 4^2)$$
$$= \frac{9800}{3}$$

$$\int 450{,}268\, e^{1{,}12567\,t}\, dt = 450{,}268\, \frac{e^{1{,}12567\,t}}{1{,}12567}$$

L'effectif de la population après trois heures est

$$400 + \int_0^3 450{,}268\, e^{1{,}12567\,t}\, dt = \left[\frac{450{,}268}{1{,}12567}\, e^{1{,}12567\,t}\right]_{t=0}^{3} + 400$$

$$\sim 10450 \text{ unités.}$$

9. (a) $$\int \left(x^3 + \frac{1}{x^2}\right)^4 dx = \int \left(x^6 + 2\,\frac{x^3}{x^2} + \frac{1}{x^4}\right)^2 dx$$

$$= \int \left(x^6 + 2x + \frac{1}{x^4}\right)^2 dx$$

$$= \int \left(x^{12} + 4x^7 + 2x^2 + 4x^2 + \frac{4}{x^3} + \frac{1}{x^8}\right) dx$$

$$= \frac{x^{13}}{13} + \frac{4}{8}x^8 + \frac{2}{3}x^3 + \frac{4}{3}x^3 + 4\,\frac{x^{-2}}{-2} + \frac{x^{-7}}{-7}$$

$$= \frac{1}{13}x^{13} + \frac{1}{2}x^8 + 2x^3 - \frac{2}{x^2} - \frac{1}{7x^7}$$

(b) $$\int \ln(x^2+1)\, dx = \int \underbrace{1}_{u'} \cdot \underbrace{\ln(x^2+1)}_{v}\, dx = x\ln(x^2+1) - \int x\, \frac{2x}{x^2+1}\, dx$$

$$= x\ln(x^2+1) - 2\int \frac{x^2}{x^2+1}\, dx$$

$$= x\ln(x^2+1) - 2\int \frac{(x^2+1)-1}{x^2+1}\, dx$$

$$= x\ln(x^2+1) - 2\left(\int dx - \int \frac{1}{x^2+1}\, dx\right)$$

$$= x\ln(x^2+1) - 2x + 2\arctan x$$

(c) $\displaystyle\int \underbrace{x^3}_{u'}\underbrace{(\ln x)}_{v}\,dx = \underbrace{\frac{x^4}{4}}_{u}\underbrace{\ln x}_{v} - \int \frac{x^4}{4}\cdot\frac{1}{x}\,dx$

$$= \frac{x^4 \ln x}{4} - \frac{1}{4}\int x^3\,dx$$

$$= \frac{x^4 \ln x}{4} - \frac{x^4}{16}\,dx$$

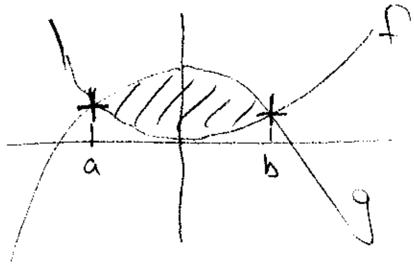
(d) $\displaystyle\int \underbrace{x^2}_{u}\underbrace{e^x}_{v'}\,dx = x^2 e^x - \int 2x e^x\,dx$

$$= x^2 e^x - 2\int x e^x\,dx$$

$$\overset{2(b)}{=} x^2 e^x - 2(x-1)e^x$$

$$= (x^2 - 2x + 2)e^x$$

10. Dessin schématique :



Déterminer a et b :

$$f(x) = g(x) \iff \frac{x^2}{4} = -\frac{x^2}{4} + x + 12$$

$$\iff \frac{x^2}{2} - x - 12 = 0$$

$$\iff x^2 - 2x - 24 = 0$$

$$\iff x = \frac{2 \pm \sqrt{4 + 96}}{2}$$

$$= \frac{2 \pm 10}{2} = 1 \pm 5$$

$$\iff x \in \{-4, 6\}$$

Donc $a = -4$, $b = 6$.

L'aire comprise entre les deux courbes est donc

$$\int_a^b (g(x)-f(x))\,dx.$$ On calcule d'abord une

primitive : $\int (g(x)-f(x))\,dx = \int\left(-\frac{x^2}{2}+x+12\right)dx$

$$= -\frac{x^3}{6}+\frac{x^2}{2}+12x$$

$$\int_a^b (g(x)-f(x))\,dx = \left[-\frac{x^3}{6}+\frac{x^2}{2}+12x\right]_{x=-4}^{6} = \frac{250}{3}$$

11. $$\int t^2e^{-t}\,dt = \int (-t)^2e^{-t}\,dt \underset{\substack{\frac{dx}{dt}=-1\\ dt=-dx}}{\overset{x=-t}{=}} -\int x^2e^x\,dx$$

$$\overset{g(d)}{=} -(x^2-2x+2)e^x = -(t^2+2t+2)e^{-t}$$

La distance parcouru entre les temps 0 et T est

$$\int_0^T t^2e^{-t}\,dt = \left[-(t^2+2t+2)e^{-t}\right]_{t=0}^{T}$$

$$= 2-(T^2+2T+2)e^{-T}$$

12. (a) $$F = \int_0^R 2\pi r\, v(r)\,dr = \int_0^R \frac{2\pi r P(R^2-r^2)}{4 v l}\,dr$$

(b) $$F = \frac{\pi P}{2 v l}\underbrace{\int_0^R (R^2r-r^3)\,dr}_{\left[R^2\frac{r^2}{2}-\frac{r^4}{4}\right]_{r=0}^{R}}$$

$$= \frac{\pi \cdot P}{2 \nu l} \cdot \frac{R^4}{4} = \frac{\pi P R^4}{8 \nu l}$$