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Übungsblatt 4 zur Einführung in die Algebra: Solutions

Aufgabe 1. Sei G eine Gruppe und $H \leqslant G$, mit $[G : H] = 2$. Zeige $H \triangleleft G$. Ist H notwendigerweise eine charakteristische Untergruppe?

Solution

Since $[G : H] = 2$, H has two left cosets and two right cosets. One coset must be H. Take $g \notin H$. Then gH and Hg are the other cosets. We have $H \cup gH = G = H \cup Hg$, and that these unions are disjoint unions. Therefore $gH = Hg$ and hence H is normal.

H need not be characteristic. Let V be the Klein four-group. Since this group is abelian, every subgroup is normal; but every permutation of the three non-identity elements is an automorphism of V, so the three subgroups of order 2 are not characteristic. Here $V = \{e,a,b,ab\}$. Consider $H = \{e,a\}$ and consider the automorphism $T(e) = e, T(a) = b, T(b) = a, T(ab) = ab$. Then $T(H)$ is not contained in H.

Aufgabe 2. Sei G eine zyklische Gruppe und $H \leqslant G$. Zeige dass H zyklisch ist.

Solution

Let G be a cyclic group and $H \leqslant G$. Without loss of generality, $H \neq \{e_G\}$.

Let g be a generator of G. We have $G = \{g^n \mid n \in \mathbb{Z}\}$. Let n be the smallest positive integer such that $g^n \in H$.

Claim: $H = \langle g^n \rangle$. We have $\langle g^n \rangle \subseteq H$ since H is a subgroup containing g^n and $\langle g^n \rangle$ is the smallest such group.

Let $h \in H$. Then $h \in G$. Let x with $h = g^x$. By the division algorithm, there exist $q,r \in \mathbb{Z}$ with $0 \leqslant r < n$ such that $x = qn + r$. Thus, $h = g^x = g^{qn+r} = g^{qn}g^r = (g^n)^q g^r$. Therefore, $g^r = h(g^n)^{-q}$. Recall that $h,g^n \in H$. Hence, $g^r \in H$. By choice of n, r cannot be positive. Thus, $r = 0$. Therefore, $h = (g^n)^q g^0 = (g^n)^q e_G = (g^n)^q = \langle g^n \rangle$. Hence, $H \subseteq \langle g^n \rangle$.

This proves the claim. It follows that every subgroup of G is cyclic.

Aufgabe 3. Sei G eine Gruppe, $N \triangleleft G$. Betrachte

$$X := \{H \mid N \leq H \leq G\} \text{ und } Y := \{J \mid J \leq G/N\}.$$

Zeige, dass

$$X \to Y,\ H \mapsto H/N$$

bijektiv ist. Seien $H,I \in X$. Zeige ferner, dass für alle $H,I \in X$ gilt:

(a) $H \leqslant I \Leftrightarrow H/N \leqslant I/N$.

(b) $H \leqslant I \Rightarrow [I : H] = [I/N : H/N]$.

(c) $H \triangleleft G \Leftrightarrow (H/N) \triangleleft (G/N)$.

Solution

Call the map in the question θ. Suppose $H, I \in X$ such that $H/N = I/N$, then for any $a \in H$ we have $aN = bN$ for some $b \in I$, and so $b^{-1}a \in N \subseteq I$. Hence $H \subseteq I$, and similarly $I \subseteq H$, so $H = I$ and θ is injective. Now suppose J is a subgroup of G/N and $\phi : G \to G/N$ by $\phi(g) = gN$. Then $\phi^{-1}(J) = \{j \in G : jN \in J\}$ is a subgroup of G (this must be proven!) containing N and $\theta(\phi^{-1}(J)) = \{jN : jN \in J\} = J$, proving that θ is bijective.

(a) $H \leq I$ iff $H/N \leq I/N$

First note that the elements of both H/N and I/N are cosets of N in G, so one must think in terms of cosets.

If $H \leq I$ then trivially $H/N \leq I/N$. Suppose now that $H/N \leq I/N$. Then $aN \in H/N \leqslant I/N$, hence $a \in \{g \in G \mid gN \in I/N\} = I$, hence $H \leq I$.

(b) $H \leq I$ implies $[I : H] = [I/N : H/N]$

Let ψ map the cosets in $I/\sim_H$ to the cosets in $\widetilde{(I/N)}/\sim_{H/N}$ by mapping the coset $bH = \widetilde{b^H}$, $b \in I$ to the coset $(bN)(H/N) = \widetilde{bH^{H/N}} = \widetilde{b^H}^{H/N}$. Then ψ is well defined and injective because:

$$\begin{aligned} b_1H = b_2H &\iff b_1^{-1}b_2 \in H \\ &\iff (b_1N)^{-1}(b_2N) = b_1^{-1}b_2N \in H/N \\ &\iff (b_1N)(H/N) = (b_2N)(H/N). \end{aligned}$$

Finally, ψ is surjective since b ranges over all of I in $(bN)(H/N)$.

(c) $H \triangleleft G$ iff $(H/N) \triangleleft (G/N)$

Suppose $H \triangleleft G$. Then for any $g \in G$ we have $(gN)(H/N)(gN)^{-1} = (gHg^{-1})/N = H/N$ and so $(H/N) \triangleleft (G/N)$.
Conversely suppose $(H/N) \triangleleft (G/N)$. Consider $\sigma\colon g \mapsto (gN)(H/N)$, the composition of the map from G onto G/N and the map from G/N onto $(G/N)/(H/N)$. $g \in \ker\sigma$ iff $(gN)(H/N) = (H/N)$ which occurs iff $gN \in H/N$ therefore $gN = aN$ for some $a \in A$. However N is contained in H, so this statement is equivalent to saying $g \in H$. So H is the kernel of a homomorphism, hence is a normal subgroup of G.

Aufgabe 4. Seien N und H Gruppen und $\varphi\colon H \to \operatorname{Aut}(N)$ ein Homomorphismus. Dann wird die Menge $N \times H$ vermöge

$$(a,b)(a',b') := (a\varphi(b)(a'),bb') \qquad (a,a' \in N,\ b,b' \in H)$$

eine Gruppe $N \rtimes_\varphi H$ und die Abbildungen $N \to N \rtimes_\varphi H$, $a \mapsto (a,1)$ und $H \to N \rtimes_\varphi H$, $b \mapsto (1,b)$ sind Einbettungen.

Solution

First we show the group axioms hold.

Associativity:

$$\begin{aligned} ((a,b)(a',b'))(a'',b'') &= (a\varphi(b)(a'), bb')(a'',b'') = (a\varphi(b)(a')\varphi(bb')(a''), bb'b'') \\ &= (a\varphi(b)(a')\varphi(b)\varphi(b')(a''), bb'b'') = (a\varphi(b)(a')\varphi(b')(a''), bb'b'') \\ &= (a,b)(a'\varphi(b')(a'''), b'b'') \end{aligned}$$

for all $(a,b), (a',b'), (a'',b'') \in N \times H$.

Neutral element:

$$(1,1)(a,b) = (1\varphi(1)(a),1 \cdot b) = (a,b)$$

for all $(a,b) \in N \times H$.

Inverse: We will show that $(a,b)(\varphi(b^{-1})(a^{-1}), b^{-1}) = (1,1) = (\varphi(b^{-1})(a^{-1}), b^{-1})(a,b)$ for all $(a,b) \in N \times H$. Let $(a,b) \in N \times H$ then

$$\begin{aligned} (a,b)(\varphi(b^{-1})(a^{-1}), b^{-1}) &= (a\varphi(b))(\varphi(b^{-1})(a^{-1})),bb^{-1}) \\ = (a(\varphi(b)(\varphi(b^{-1}))a^{-1}),1) &= (a(\varphi(bb^{-1}))a^{-1}),1) = (a\varphi(1)a^{-1},1) = (aa^{-1}, 1) = (1,1). \end{aligned}$$

The other calculation is similar.

Finally we show that the two maps in the question are embeddings:

$$(a,1)(a',1) = (a\varphi(1)(a'),1) = (aa',1)$$

and

$$(1,b)(1,b') = (1\varphi(b)1,bb') = (1,bb')$$

for all $a,a' \in N$ and $b,b' \in H$, and therefore we our maps have kernel $\{1,1\}$.