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Übungsblatt 5 zur Einführung in die Algebra: Solutions

Aufgabe 1. Sei R ein kommutativer Ring, der genau 3 Ideale hat, $(0), I$ und R. Zeige, dass:

(1) $a - 1 \in R^\times$ für alle $a \in I$.

(2) $ab = 0$ für alle $a,b \in I$.

Finde ein Beispiel solches Ringes.

Solution

Let J be the ideal generated by $a-1$. We cannot have that $J = (0)$ as $a \neq 1$, since $1 \notin I$ as $I \neq R$. We cannot have that $J = I$ as this would imply that $a - (a-1) = 1 \in I$ and $I = R$, which is again false. So, $J = R$ and hence $a - 1$ must be a unit

For (2), we again note that the ideal generated by ab, which we call H, is either (0), I or R. We cannot have $H = R$ as $H \subseteq I \neq R$.

Assume $H = I$ then $a = (ab)c$ for some $c \in R$ as $a \in I$, hence $a(bc-1) = 0$. But $bc \in I$, hence $bc - 1$ is a unit, and we cannot have $au = 0$ for any $a \in R$ and $u \in R^\times$ unless $a = 0$, and in that case $ab = 0$. Then $(0) = H \neq I$, a contradiction, so this case cannot occur.

Hence, $H = (0)$, and hence $ab = 0$.

Finally, $R = \mathbb{Z}/(4)$ has only 3 ideals, (0), $(\overline{2})$ and the whole ring. This is clear, as if an ideal contains $\overline{3}$ or $\overline{1}$ then it must contain the whole ring.

Aufgabe 2. Sei R eine kommutativer Ring und $I,J \subset R$ Ideale von R. Sei

$$I + J := \{i + j \mid i \in I \text{ und } j \in J\}$$

und

$$IJ := \{i_1 j_1 + \ldots + i_n j_n \mid i \in I \text{ und } j \in J, i = 1, \ldots, n\}.$$

Zeige, dass $I + J = R \Rightarrow I \cap J = IJ$.

Was ist wenn R nicht kommutativ ist?

(Hinweis: betrachte die 2×2 invertierbare untere Dreiecksmatrizen.)

Solution

We claim that $(I+J)(I \cap J) \subseteq IJ$. This is true, because $(I+J)(I\cap J) = I(I \cap J) + J(I \cap J) \subseteq IJ + IJ = IJ$.

However, $I + J = R$, so, the above identity reduces to $I \cap J \subseteq IJ$.

$IJ \subseteq I \cap J$ is straightforward, since IJ consists of all finite sums of products of elements of I by elements of J. Each of these elements are in both I and J since ideals are closed under (ideal) addition and ring multiplication.

In the non-commutative case, this is not true. Let R be the right of lower triangular matrices over a ring with more than 1 element. Let $I \subseteq R$ be the ideal generated by $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $J \subseteq R$ be the ideal generated by $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Then clearly $I + J = R$ as it contains the identity.

$IJ = \{0\}$ as $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a' & 0 \\ c' & b' \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = 0$ for all $a,b,a',b',c' \in R$,

Finally we have $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in J$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in I$, so $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in I \cap J$ for all $c \in R$. So $IJ \neq I \cap J$.

Aufgabe 3. Betrachte den $\mathbb{R}$-Vektorraum $\mathbb{R}^{\mathbb{N}}$ aller reellen Folgen und dessen Endomorphismenring $A := \text{End}(\mathbb{R}^{\mathbb{N}})$. Finde ein $f \in A$, welches linksinvertierbar ist (d.h. es gibt $g \in A$ mit $gf = 1_A$), aber nicht rechtsinvertierbar ist.

Solution

Let $f \in \text{End}(\mathbb{R}^{\mathbb{N}})$ be given by

$$f(a_0,a_1,a_2,\ldots) = (0,a_0,a_1,\ldots).$$

Then f has a left inverse given by

$$g(a_0,a_1,a_2,\ldots) = (a_1,a_2,a_3\ldots).$$

Assume f has a right inverse h. Then we must have $h = g$ as $g = g(fh) = (gf)h = h$. However $fg \neq 1$, so f has no right inverse.

Aufgabe 4. Betrachte den Ring $\mathbb{R}^{\mathbb{R}}$ aller Funktion $\mathbb{R} \to \mathbb{R}$ (mit punktweiser Addition und Multiplikation) und dem Einsetzungshomomorphismus

$$\varphi : \mathbb{R}[X,Y] \to \mathbb{R}^{\mathbb{R}}$$

$$f \mapsto f(\cos,\sin).$$

Zeige $\ker(\varphi) = (X^2 + Y^2 - 1)$.

(**Hinweis**: Betrachte zunächst Polynome der Form $g + Yh$ mit $g,h \in \mathbb{R}[X]$)

Bemerkung: Die Elemente vom $\text{im}(\varphi)$ nennt man *trigonometrische Polynome*.

Solution

Firstly note that $(X^2 + Y^2 - 1) \subseteq \ker(\varphi)$ as $\cos^2 + \sin^2 = 1$.

Now assume that $f \in \ker(\varphi)$. Assume further, as in the hint, $f = g + Yh$ for $g,h \in \mathbb{R}[X]$. Then

$$0 = f(\cos,\sin) = g(\cos) + \sin h(\cos).$$

However, since $\sin(x) = \sin(-x)$ and $\cos(x) = \cos(-x)$ for all $x \in \mathrm{R}$, we have

$$0 = f(\cos,\sin) = g(\cos) - \sin h(\cos).$$

Adding these expressions together gives $g(\cos) = 0$ and hence $g = 0$ as $g(\cos(x)) = 0$ for all $x \in \mathrm{R}$, and therefore $g(Y) = 0$ for all $Y \in [-1,1]$. Similarly we get $h = 0$, so $f = 0$.

Now assume that $f \in \ker(\varphi)$ but $f \notin (X^2 + Y^2 - 1)$. We may write $f = g_0 + g_1 X + g_2 X^2 + \cdots + g_n X^n$ for some $g_i \in \mathbb{R}[Y]$. Substituting X^2 for $1 - Y^2$ in this expression as many times as necessary results in a polynomial of the form $g + Yh$ with $g,h \in \mathbb{R}[X]$, which is congruent to f modulo $(X^2 + Y^2 - 1) \subseteq \ker(\varphi)$. Thus we assume that f is of this form. But we have already shown that in this case $f = 0$. Hence $\ker(\varphi) \subseteq (X^2 + Y^2 - 1)$, and hence the result.