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Übungsblatt 10 zur Einführung in die Algebra: Solutions

Aufgabe 1. Sei G eine Gruppe der Ordnung p^2q, für zwei Primzahlen $p \neq q$. Zeige, dass G eine p–Sylowgruppe oder eine q–Sylowgruppe enthält, die ein Normalteiler ist.

Solution

Since all p or q-Sylow subgroups are conjugate, if a Sylow p or q-subgroup is unique then it must be normal.

Assume $p > q$. Then the number of p-Sylow subgroups is $1 + pk$ for some $k \in \mathbb{N}_0$ and divides p^2q. So $k = 0$ and hence there is a unique p-Sylow subgroup.

Assume $p < q$. Then the number of q-Sylow subgroups is $1 + qn$ for some $n \in \mathbb{N}_0$, and must divide p^2. Either $n = 0$ (and hence there is a unique q-Sylow subgroup), or $1 + nq = p^2$. In the latter case, this gives $(q-1)p^2$ elements of order q in G, as any two q–Sylow subgroups meet only in 1. G also contains at least one p-Sylow subgroup, which is of order p^2 and only intersects the q-Sylow subgroups at the identity. Since there are only $p^2q - p^2(q-1) = p^2$ elements that are not of order q, these elements must form a unique p-Sylow subgroup of order p^2.

Aufgabe 2. Sei G eine Gruppe und $H \leqslant G$. Sei

$$N_G(H) := \{g \in G \,|\, gHg^{-1} = H\}.$$

Wir nennen $N_G(H)$ den *Normalisator* von H in G.

Zeige, dass

(1) $N_G(H) \leqslant G$.

(2) $H \lhd N_G(H)$.

(3) $H \lhd G \Leftrightarrow N_G(H) = G$.

(4) es eine Bijektion zwischen den Mengen $\{gN_G(H) \mid g \in G\}$ und $\{gHg^{-1} \mid g \in G\}$ gibt.

Solution

Let G act by conjugation on the set of subgroups of G. Then $N_G(H)$ is the stabilizer of H (this shows (1)) and $\{gHg^{-1} \mid g \in G\}$ is the orbit of H. By the orbit-stabilizer theorem there is a bijection between the set of left cosets of the stabilizer and the orbit. This shows (4).

If $H \lhd G$, it's orbit under this action consists of only one point and hence $N_G(H) = G$. Similarly if $N_G(H) = G$, then it's orbit is only point point, and $H \lhd G$. This shows (3).

Clearly $hHh^{-1} = H$ for all $h \in H$, so $H \leqslant N_G(H)$. Let $x \in N_G(S)$. Then $xHx^{-1} = H$ by definition. So $H \lhd N_G(H)$. This shows (2).

Aufgabe 3. Sei G eine endliche Gruppe und sei $H \leqslant G$. Sei $\tau : G \times X \to X$ eine transitive Gruppenwirkung und $x \in X$.

(i) Zeige, dass die Einschränkung von τ auf $H \times X$ genau dann transitiv ist, wenn $G = HG_x$, wobei $G_x = \{g \in G \,|\, \tau(g,x) = x\}$ und $HG_x = \{hg \mid g \in G_x, h \in H\}$.

(ii) Sei $M \lhd G$ und P eine p–Sylowgruppe von M. Zeige, dass $G = MN_G(P)$.

Solution

(i) Suppose $G = HG_x$. Since the group action is transitive, for each $x,y \in X$, there is a $g \in G$ such that $\tau(g,x) = y$. We can write $g = hg'$ for $h \in H$ and $g' \in G_x$. Then $y = \tau(g,x) = \tau(hg',x) = \tau(h,\tau(g',x)) = \tau(h,x)$ as $g' \in G_x$. Hence H acts transitively.

Conversely, let $g \in G$. If H acts transitively, then there exists an $h \in H$ such that $\tau(g,x) = \tau(h,x)$. Then $h^{-1}g \in G_x$ and hence the result.

(ii) Let X be the set of p-Sylow subgroups of M and $\tau : G \times X \to X$ be the action given by conjugation. This is well defined as M is a normal subgroup of G. By the Sylow theorem, the restriction of τ to $H \times X$ is transitive. For $P \in X$, we have that $G_P = \{g \in G \mid gPg^{-1} = P\} = N_G(P)$. Hence we apply the first part of the question to get $G = MN_G(P)$.

Aufgabe 4. Sei G eine Gruppe der Ordnung pq, für Primzahlen $p < q$. Zeige, dass G zyklisch ist, wenn p nicht $(q-1)$ teilt.

Solution

Then the number of q-Sylow subgroups is $1 + qn$ for some $n \in \mathbb{N}_0$and must divide p. Hence $n = 0$ and G contains a unique q-Sylow subgroup, which we call Q.

The number of p-Sylow subgroups is $1 + pk$ for some $k \in \mathbb{N}_0$ and divides q. Hence either $k = 0$ or $1 + pk = q$. But $1 + pk = q$ implies that p divides $q - 1$, contradicting our assumptions. Hence $k = 0$, so there is also a unique p-Sylow subgroup, which we call P.

Since P and Q are unique, we have that elements in $G \setminus \{P \cup Q\}$ do not have prime order, otherwise they would be contained in another p or q-Sylow subgroup. Hence they must have order pq. Such an element must exist as $pq > p + q - 1$. G must be generated by this element, and hence G is cyclic.

Aufgabe 5. Seien $p,q \in \mathbb{N}_0$ ungerade und prim (möglicherweise gleich). Sei G eine Gruppe der Ordnung $2pq$. Zeige, dass G eine eindeutige p–Sylowgruppe oder eine eindeutige q–Sylowgruppe (oder beide) enhält.

Solution

If $p = q$ then G has order $2p^2$. Therefore a p-Sylow group has index 2 and is therefore a normal subgroup, and hence unique.

Assume now that $p \neq q$. The number of p-Sylow groups in G is $1 + pn$ for some $k \in \mathbb{N}_0$ and the number of q-Sylow subgroups in G is is $1 + qn$ for some $n \in \mathbb{N}_0$.

Assume $n,k \geqslant 1$. Then $1+pk \geqslant p+1$ and $1+qn \geqslant q+1$, so there are at least $(p-1)(p+1) = p^2-1$ elements of order p in G (as each p-Sylow group has $p-1$ elements of order p and the intersection of each pair of p-Sylow groups is $\{1\}$) and at least $q^2 - 1$ elements of order q.

There is also at least one element of order 2 (see sheet 2, question 2), and the trivial element. This implies that the order of G is $|G| = 2pq \geqslant (p^2-1) + (q^2-1) + 2 = p^2 + q^2$. Rearranging, gives $(p-q)^2 \leqslant 0$, and hence $p = q$, a contradiction. Hence either n or k (or both) must equal 1, i.e. G must contain a unique p-Sylow subgroup or a unique q-Sylow subgroup (or both!).

Aufgabe 6. Sei K ein Körper. Zeige, dass die Gruppe von invertierbaren oberen 3×3–Dreiecksmatrizen über K auflösbar ist.

Solution

Let G be the group of invertible upper triangular 3x3 matrices.

For any invertible upper triangular matrix A, the entries on the main diagonal are non-zero, and the entries on the main diagonal of A^{-1} must therefore be the inverses of the entries of the main diagonal of A. So the entries on the main diagonal of $ABA^{-1}B^{-1}$, for two invertible upper triangular matrices A and B, are all 1.

Let $G^{(1)} = G' = \langle ABA^{-1}B^{-1} \mid x,y \in G\rangle$, the commutator subgroup of G. Consider elements of $G^{(2)} = (G^{(1)})'$, all of which have all 1's on the main diagonal. A simple calculation shows that if $A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ then $A^{-1} = \begin{pmatrix} 1 & -a & y \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$ for some $y \in K$. Direct computation then shows that every element in $G^{(2)}$ is of the form $C = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for some $x \in K$.

Calculating again shows that $C^{-1} = \begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and that $(G^{(2)})' = \{e\}$, and hence G is solvable.

Aufgabe 7. Sei K ein Körper. Zeige, dass $\mathrm{GL}_2(K)' = \mathrm{SL}_2(K)$.

Hinweis: Betrachte

$$\left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right], \left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}\right], \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right] \in \mathrm{GL}_2(K)'.$$

Solution

It's clear that $[A,B]$ has determinant 1 for all $A,B \in \mathrm{GL}_2(F)$, hence $\mathrm{GL}_2(K)' \subseteq \mathrm{SL}_2(K)$.

Now, consider the commutators from the hint. Direct calculation shows that

$$\left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 1-y \\ 0 & 1 \end{pmatrix},$$

$$\left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$$

and

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix}.$$

Hence, for all $x,y \in F$ matrices of the form

$$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \text{ and } \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix}$$

are elements of $\mathrm{GL}_2(K)'$.

Now, take a matrix in $\mathrm{SL}_2(F)$, say $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a \neq 0$. We multiply it on the right by $\begin{pmatrix} 1 & -b/a \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(K)'$ to reduce to the case of $b = 0$. Given a matrix $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$, we multiply it on the left by $\begin{pmatrix} 1 & 0 \\ -c/a & 1 \end{pmatrix} \in \mathrm{GL}_2(K)'$ to see that we may assume $c = 0$, and we are left with a form $\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$ (as the determinant must be 1), which is a commutator by the above. This shows that any matrix in $\mathrm{SL}_2(K)$ of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a \neq 0$ is in the commutator subgroup.

If $a = 0$ the we have $\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$ and $c \neq 0$. Multiply on the left by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(K)'$ to get back to the case $a \neq 0$.