Universität Konstanz
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Real Algebraic Geometry I – Exercise Sheet 5

Exercise 1 (6P). Let $(K, \leq)$ be an ordered field.

(a) Show $\sigma(f) \geq \sigma((1+X)f)$ for all $f \in K[X]$.

(b) Suppose now that K is real closed. Find the number of positive and the number of negative roots in K of the polynomial $f := X^5 - X^4 + 3X^3 + 9X^2 - X + 5 \in K[X]$.

Exercise 2 (3P). Let $(K, \leq)$ be an ordered field. Show the following:

(a) Suppose $a_0, \dots, a_d \in K$, $a_d \neq 0$ and $x \in K$ with $|x| > \max\left\{1, \frac{|a_0|}{|a_d|} + \dots + \frac{|a_{d-1}|}{|a_d|}\right\}$. Then $|a_d x^d| > |\sum_{i=0}^{d-1} a_i x^i|$

(b) Let $f \in K[X]$. For $f(X+a)$ and $f(X-a)$, the evaluations of f in $X+a$ and $X-a$, respectively, determine $\sigma(f(X+a))$ and $\sigma(f(X-a))$ if $a \in K$ is large.

Exercise 3 (4P). Suppose $0 \neq f \in \mathbb{R}[X_1, \dots, X_n]$ and $r > 0$. Show

$$\exists x \in \mathbb{R}^n : \forall y \in \mathbb{R}^n : (f(y) = 0 \implies \|x - y\| \geq r).$$

Exercise 4 (4P). Consider the following variant of Descartes' Rule of signs. Let $k \in \mathbb{N}$, $c_1, \dots, c_k \in \mathbb{R}^\times$ and $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ with $\alpha_1 < \dots < \alpha_k$ and consider the function

$$f \colon \mathbb{R}_{>0} \to \mathbb{R}, x \mapsto c_1 x^{\alpha_1} + \dots + c_k x^{\alpha_k}.$$

For $a \in \mathbb{R}_{>0}$, define

$$\mu(a, f) := \sup\{m \in \mathbb{N}_0 \mid f^{(0)}(a) = \dots = f^{(m-1)}(a) = 0\} \in \mathbb{N}_0 \cup \{\infty\}.$$

Define furthermore

$$\mu(f) := \sum_{a \in \mathbb{R}_{>0}} \mu(a, f) \in \mathbb{N}_0 \cup \{\infty\},$$

where the sum should be understood as infinity if one of the terms is infinite or if it has infinitely many non-zero terms, and

$$\sigma(f) = \#\{i \in \{1, \dots, k-1\} \mid \operatorname{sgn}(c_i) \neq \operatorname{sgn}(c_{i+1})\} \in \{0, \dots, k-1\}.$$

Show $\mu(f) \leq \sigma(f) < k < \infty$.

Hint. Induction on k. If $\mu(f) \geq 1$, find i with $\operatorname{sgn}(c_i) \neq \operatorname{sgn}(c_{i+1})$, divide by $x \mapsto x^{\alpha_i}$ in order to assume $\alpha_i = 0$ and show then $\sigma(f') = \sigma(f) - 1$ by a case distinction and $\mu(f') \geq \mu(f) - 1$.

Please submit until Thursday, December 1, 2016, 11:44 in the box named RAG I, Number 10, near to the room F411.