Universität Konstanz
Department of Mathematics and Statistics
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Real Algebraic Geometry I – Exercise Sheet 9

Exercise 1 (4P). Give an algorithm deciding if a polynomial $f \in \mathbb{Q}[X, Y]$ has only finitely many roots in $\mathbb{R}^2$.

Exercise 2 (4P). Let R be a real closed field and $n \in \mathbb{N}_0$. Show that a semialgebraic set $A \subseteq R^n$ has nonempty interior $\mathring{A}$ if and only if A is Zariski-dense in R^n (i.e., if no polynomial $f \in R[X_1, ..., X_n] \setminus \{0\}$ vanishes on A).

Exercise 3 (8P). Show that the following sets are not K-semialgebraic:

(a) the garden fence $\{(x, y) \in \mathbb{R}^2 \mid y \geq 0,\ y \leq |\lfloor x \rfloor - x + \frac{1}{2}| + 10\}$ where $K := \mathbb{R}$,

(b) $\{(x, 2^x) \mid x \in \mathbb{R}\}$ where $K := \mathbb{Q}$,

(c) the set of all *infinitesimal* elements in an arbitrary fixed non-archimedean real closed field R where $K := R$, and

(d) the set $\{(x, y, z) \in R^3_{>0} \mid \forall n \in \mathbb{N} : x \geq yn \geq zn^2\}$ for an arbitrary fixed non-archimedean real closed extension field R of $K := \mathbb{R}$.

Bonus exercise (4BP). Let R be a real closed field and A a finitely generated R-algebra. Suppose there exists an algebra homomorphism $\varphi\colon A \to S$ where S into a real closed extension field S of R.

(a) Show that there exists also an algebra homomorphism $A \to R$.

(b) Give a counterexample to (a) in the case where one drops the requirement that R is real closed.

Hint: For (a), find an ideal I with $\psi\colon A \overset{\cong}{\to} R[\underline{X}]/I$ and analyze the algebra homomorphism $\gamma\colon R[\underline{X}] \to R_1, f \mapsto \varphi(\psi^{-1}(\overline{f}^I))$ which is a point evaluation.

Please submit until Thursday, January 12, 2017, 11:44 in the box named RAG I, Number 10, near to the room F411.