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Geometry of Linear Matrix Inequalities – Exercise Sheet 1

Exercise 1 (4P) We call a polynomial $f \in \mathbb{R}[\underline{X}]$ *strictly quasi-concave* in a point $x \in \mathbb{R}^n$ if

$$-v^T \operatorname{Hess}(g)(x)v > 0$$

for all $v \in \mathbb{R}^n \setminus \{0\}$ with $\nabla g(x)^T v = 0$. Let $f \in \mathbb{R}[\underline{X}]$ be strictly quasi-concave on $\mathbb{R}^n$. For arbitrary $y \in \mathbb{R}$, show:

(a) $S_y := \{x \in \mathbb{R}^n \mid f(x) \geq y\}$ is convex.

(b) All boundary points of S_y are extreme points of S_y.

Exercise 2 (6P) Let $S \subseteq \mathbb{R}^n$ compact and $f \in \mathbb{R}[\underline{X}]$ strictly quasi-concave on S. Show that there exists $M \in \mathbb{N}$ and an $\varepsilon > 0$ such that $-\operatorname{Hess}(f)(x) + M\nabla f(x)\nabla f(x)^T - \varepsilon I$ is pd for all $x \in S$.

Exercise 3 (6P) Set $\mathfrak{m} := (X_1, \dots, X_n) \subseteq \mathbb{R}[\underline{X}]$ and $u := X_1^2 + \ldots + X_n^2$. Suppose that $g_1, \dots, g_\ell \in \mathfrak{m}$ are strictly quasi-concave at 0 and satisfy $g_1(0) = \ldots = g_\ell(0) = 0$. Moreover, suppose that $h_1, \dots, h_k \in \mathbb{R}[\underline{X}]$ satisfy $h_1(0) > 0, \dots, h_k(0) > 0$. Consider the quadratic module M generated by

$$g_1, \dots, g_\ell, h_1, \dots, h_k$$

and suppose that

$$S := \{x \in \mathbb{R}^n \mid g_1(x), \dots, g_\ell(x), h_1(x), \dots, h_k(x) \geq 0\}$$

is convex with nonempty interior. Fix $v \in \mathbb{R}^n \setminus \{0\}$ and define

$$\varphi\colon \mathfrak{m}^2 \to \mathbb{R},\ f \mapsto v^T \operatorname{Hess}(f)(0)v.$$

Show that φ is a state of $\left(\mathfrak{m}^2, M \cap \mathfrak{m}^2, \frac{u}{\varphi(u)}\right)$.

Exercise 4 (4P) Show that the quadratic module generated by $X - 1$, $Y - 1$ and $1 - XY$ in $\mathbb{R}[X, Y]$ is not Archimedean.

Exercise 5 (4P) Find an Archimedean quadratic modules $M \subseteq \mathbb{R}[\underline{X}]$ and $f \in M$, such that f is not a unit of $M \cap (f)$ in (f).

Please submit until Tuesday, July 11, 2017, 9:55 in the box named RAG II near to the room F411.