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Geometry of linear matrix inequalities - Exercise sheet 3

Exercise 1 (10P) Set $g := 1 - X^4 - Y^4$ and $S := \{(x, y) \in \mathbb{R}^2 \mid g(x) \geq 0\}$.

(a) Show that S is convex.

(b) Show that S is a not a spectrahedron.

(c) Show that every $f \in \mathbb{R}[X, Y]_1$ with $f \geq 0$ on S is an element of $M_4(g)$.

(d) Find a spectrahedron $S' \subseteq \mathbb{R}^4$ such that

$$S = \{(x, y) \in \mathbb{R}^2 \mid \exists s, t \in \mathbb{R} : (x, y, s, t) \in S'\}.$$

Hint: For (c), use Hilbert's 1888 Theorem 7.5.10 and Lagrange multipliers.

Exercise 2 (10P) Let $n \in \mathbb{N}$, $g \in \mathbb{R}[\underline{X}]$ and $x \in \mathbb{R}^n$ such that $g(x) = 0$ and $\nabla g(x) \neq 0$. Suppose $v_1, \ldots, v_n$ form a basis of $\mathbb{R}^n$, U is an open neighborhood of 0 in $\mathbb{R}^{n-1}$ and $\varphi\colon U \to \mathbb{R}$ is smooth and satisfies $\varphi(0) = 0$ and

$$(*) \qquad g(x + \xi_1 v_1 + \ldots + \xi_{n-1} v_{n-1} + \varphi(\xi) v_n) = 0$$

for all $\xi = (\xi_1, \ldots, \xi_{n-1}) \in U$. Then the following hold:

(a) $(\nabla g(x))^T v_1 = \ldots = (\nabla g(x))^T v_{n-1} = 0 \iff \nabla \varphi(0) = 0$

(b) If $\nabla \varphi(0) = 0$ and $(\nabla g(x))^T v_n > 0$, then

$$g \text{ is strictly quasiconcave at } x \iff \operatorname{Hess} \varphi(0) \succ 0.$$

Exercise 3 (10P) Let $n \in \mathbb{N}$, $g \in \mathbb{R}[\underline{X}]$ and $x \in \mathbb{R}^n$ such that $g(x) = 0$. Let V be a neighborhood of x and $v_1, \ldots, v_n$ be a basis of $\mathbb{R}^n$. the following are equivalent:

(a) $\nabla g(x) v_n > 0$

(b) $g(x + \lambda v_n) > 0$ for all small enough $\lambda \in \mathbb{R}_{>0}$.

(c) $x + \lambda v_n \in (S(g) \setminus Z(g)) \cap V$ for all small enough $\lambda \in \mathbb{R}_{>0}$.

If the equivalent conditions (a)–(c) are satisfied, then the following conditions are also equivalent:

(e) g is strictly quasiconcave at x.

(f) There is an open neighborhood U of 0 in $\mathbb{R}^{n-1}$ and a smooth function $\varphi\colon U \to \mathbb{R}$ such that $\varphi(0) = 0$, $\nabla\varphi(0) = 0$, $\operatorname{Hess}\varphi(0) \succ 0$ and

$$(*) \qquad g(x + \xi_1 v_1 + \ldots + \xi_{n-1} v_{n-1} + \varphi(\xi) v_n) = 0$$

for all $\xi = (\xi_1, \ldots, \xi_{n-1}) \in U$.

(g) Condition (f) holds with $(*)$ replaced by

$$(**) \qquad x + \xi_1 v_1 + \ldots + \xi_{n-1} v_{n-1} + \varphi(\xi) v_n \in Z(g) \cap V.$$

If the equivalent conditions (e)–(g) are satisfied, then $\nabla g(x) v_i = 0$ for all $i \in \{1, \ldots, n-1\}$.

Please submit until Tuesday, July 25, 2017, 9:55 in the box named RAG II near to the room F411.