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Summer Term 2017

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Real Algebraic Geometry II – Exercise Sheet 6

Exercise 1 (8P) Let K be a subfield of $\mathbb{R}$ and V be a K-vector space. We call a map $p\colon V \to \mathbb{R}_{\geq 0}$ a *seminorm* on V if

$$\begin{aligned} &\forall x, y \in V : p(x+y) \leq p(x) + p(y) \qquad \text{and} \\ &\forall \lambda \in K : \forall x \in V : p(\lambda x) = |\lambda| p(x). \end{aligned}$$

Now let V be a topological K-vector space.

(a) Show that any neighborhood U of 0 in V is *absorbing*, i.e.,

$$V = \bigcup_{\lambda \in K_{>0}} \lambda U.$$

(b) Show that for any convex balanced neighborhood U of 0 in V, the function

$$p_U\colon V \to \mathbb{R},\ x \mapsto \inf\{\lambda \in K_{>0} \mid x \in \lambda U\}$$

is a seminorm on V with

$$U^\circ = \{x \in V \mid p(x) < 1\} \subseteq U \subseteq \{x \in V \mid p(x) \leq 1\} = \overline{U}.$$

(c) Prove that the correspondence

$$\begin{aligned} U &\mapsto p_U \\ \{x \in V \mid p(x) < 1\} &\leftarrow\!\shortmid p \end{aligned}$$

defines a bijection between the set of all open convex balanced neighborhoods of 0 in V and the set of all continuous seminorms on V.

Exercise 2 (6P) Let K be a subfield of $\mathbb{R}$ and V be a K-vector space. If P is a set of seminorms on V, we denote by $\mathscr{O}_P$ the topology on V generated by the sets

$$\{x \in V \mid p(x-y) < \varepsilon\} \qquad (p \in P, y \in V, \varepsilon > 0).$$

We call P *separating* if for all $x \in V \setminus \{0\}$ there exists $p \in P$ such that $p(x) \neq 0$. Now let $\mathscr{O}$ be a topology on the set V. Show that the following are equivalent:

(a) $(V, \mathscr{O})$ is a locally convex K-vector space.

(b) There exists a separating set P of seminorms on V such that $\mathscr{O} = \mathscr{O}_P$.

Exercise 3 (6P) Let $n \in \mathbb{N}_{\geq 2}$. The one-dimensional affine subspaces of $\mathbb{R}^n$ are called *lines*.

(a) Show that the following defines a topology on $\mathbb{R}^n$: A set $A \subseteq \mathbb{R}^n$ is open if and only if for every line G the intersection $G \cap A$ is open in G with respect to the topology induced on G by $\mathbb{R}^n$.

(b) Is the addition $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,\ (x,y) \mapsto x+y$ continuous with respect to this topology?

(c) Is the scalar multiplication $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n,\ (\lambda, x) \mapsto \lambda x$ continuous with respect to this topology?

Exercise 4 (8P+2BP) This exercise should be done without using the separation theorems in §7 of the lecture notes. Let A be a nonempty closed convex subset of $\mathbb{R}^n$.

(a) Show that for each $x \in \mathbb{R}^n$, there is a unique $\pi(x) := y$ in $\mathbb{R}^n$ such that

$$\|x - y\| < \|x - z\|$$

for all $z \in A \setminus \{x\}$.

(b) Show that the corresponding map $\pi\colon \mathbb{R}^n \to A$ is contractive, i.e.,

$$\|\pi(x) - \pi(y)\| \leq \|x - y\|$$

for all $x, y \in \mathbb{R}^n$.

(c) Let $A \subseteq \mathbb{R}^n$ be closed and convex and $x \in \mathbb{R}^n \setminus A$. Show that there is an $\mathbb{R}$-linear $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ with $\varphi(x) < \varphi(a)$ for all $a \in A$.

(d) (Bonus) Let $C \subseteq \mathbb{R}^n$ be compact and convex. Let $A \subseteq \mathbb{R}^n$ be closed and convex such that $A \cap C = \emptyset$. Prove: There are $\varepsilon > 0$ and an $\mathbb{R}$-linear function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ such that $\varphi(x) + \varepsilon \leq \varphi(a)$ for all $a \in A$ and $x \in C$.

Please submit until Tuesday, June 6, 2017, 9:55 in the box named RAG II near to the room F411.