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Real Algebraic Geometry II – Exercise Sheet 10

Exercise 1 (4P) Let $f \in \mathbb{R}[\underline{X}]$ be homogeneous and positive definite. Show that there exists $k \in \mathbb{N}$ such that $(\sum_{i=1}^{n} X_i^2)^k f \in \sum \mathbb{R}[\underline{X}]^2$.

Hint: Try to imitate the proof of Example 8.2.15. Use the substitution $X_i \mapsto \frac{X_i}{\sqrt{X_1^2+...+X_n^2}}$ and calculate in the field $\mathbb{R}(\underline{X})\left(\sqrt{X_1^2+...+X_n^2}\right)$.

Exercise 2 (4P) Let K be a subfield of $\mathbb{R}$ equipped with the induced order and let A be a commutative ring containing K. Suppose T is a preorder or Archimedean semiring of A, $K_{\geq 0} \subseteq T$ and $M \subseteq A$ is an Archimedean T-module of A. Set

$$S := \{x \in \mathbb{R}^n \mid \forall f \in M : \; f(x) \geq 0\}.$$

Suppose that there are $a_1, ..., a_m \in A$ and $f_1, ..., f_m \in T$ such that $f = \sum_{i=1}^{m} f_i a_i$, $f \geq 0$ on S and $a_i(x) > 0$ for all $x \in S$ with $f(x) = 0$. Show that $f \in M$.

Exercise 3 (4P) Let A be a commutative ring. We say that a preorder T is *saturated* if it is an intersection of prime cones.

(a) Show that a preorder T of A is saturated if and only if for all $f \in A$, $m \in \mathbb{N}$, $s, t \in T$ the equality $sf = t + f^{2m}$ implies $f \in T$.

(b) Suppose now that $A = \mathbb{R}[\underline{X}]$ and that T is finitely generated. Define

$$S = \{x \in \mathbb{R}^n \mid \forall t \in T : \; t(x) \geq 0\}.$$

Show that T is saturated if and only if every polynomial $f \in \mathbb{R}[\underline{X}]$ fulfilling $f \geq 0$ on S is contained in T.

Exercise 4 (4P) Let A be a commutative ring containing $\mathbb{Q}$. Show that an Archimedean preorder T of A is saturated if and only if $T_{\mathfrak{m}} := (A \setminus \mathfrak{m})^{-2} T$ is saturated in

$$A_{\mathfrak{m}} := (A \setminus \mathfrak{m})^{-1} A$$

for every maximal ideal $\mathfrak{m} \subseteq A$.

Exercise 5 (4P) Suppose that T is an Archimedean preorder of $\mathbb{R}[\underline{X}]$,

$$S = \{x \in \mathbb{R}^n \mid \forall g \in T : \; g(x) \geq 0\}$$

and $f \in T + (f^2)$ satisfies $f \geq 0$ on S. Show that $f \in T$.

Hint: Write $f = t + hf^2$ with $t \in T$ and $h \in \mathbb{R}[\underline{X}]$. Work with $I := (t, f^2)$.

Exercise 6 (6P) Decide if the preorder generated by

(a) $1 - X^2$

(b) $-1 + X^2$

in $\mathbb{R}[X]$ is saturated.

Please submit until Tuesday, July 4, 2017, 9:55 in the box named RAG II near to the room F411.