Spectrahedral relaxations of hyperbolicity cones

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POEMA Workshop 3

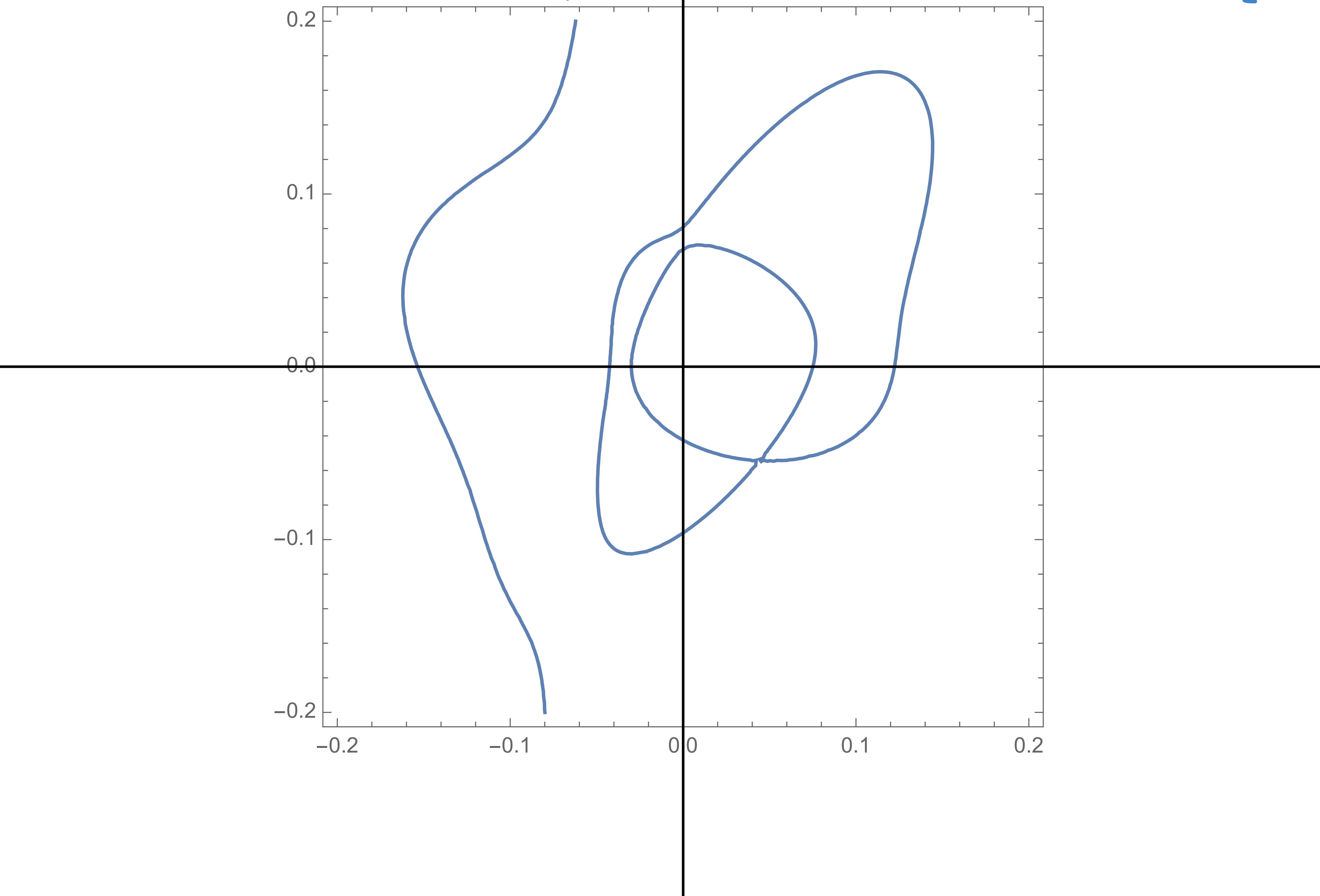
Polynomial Optimization, Efficiency through Moments and Algebra

online due to pandemic
originally planned at INRIA Sophia Antipolis

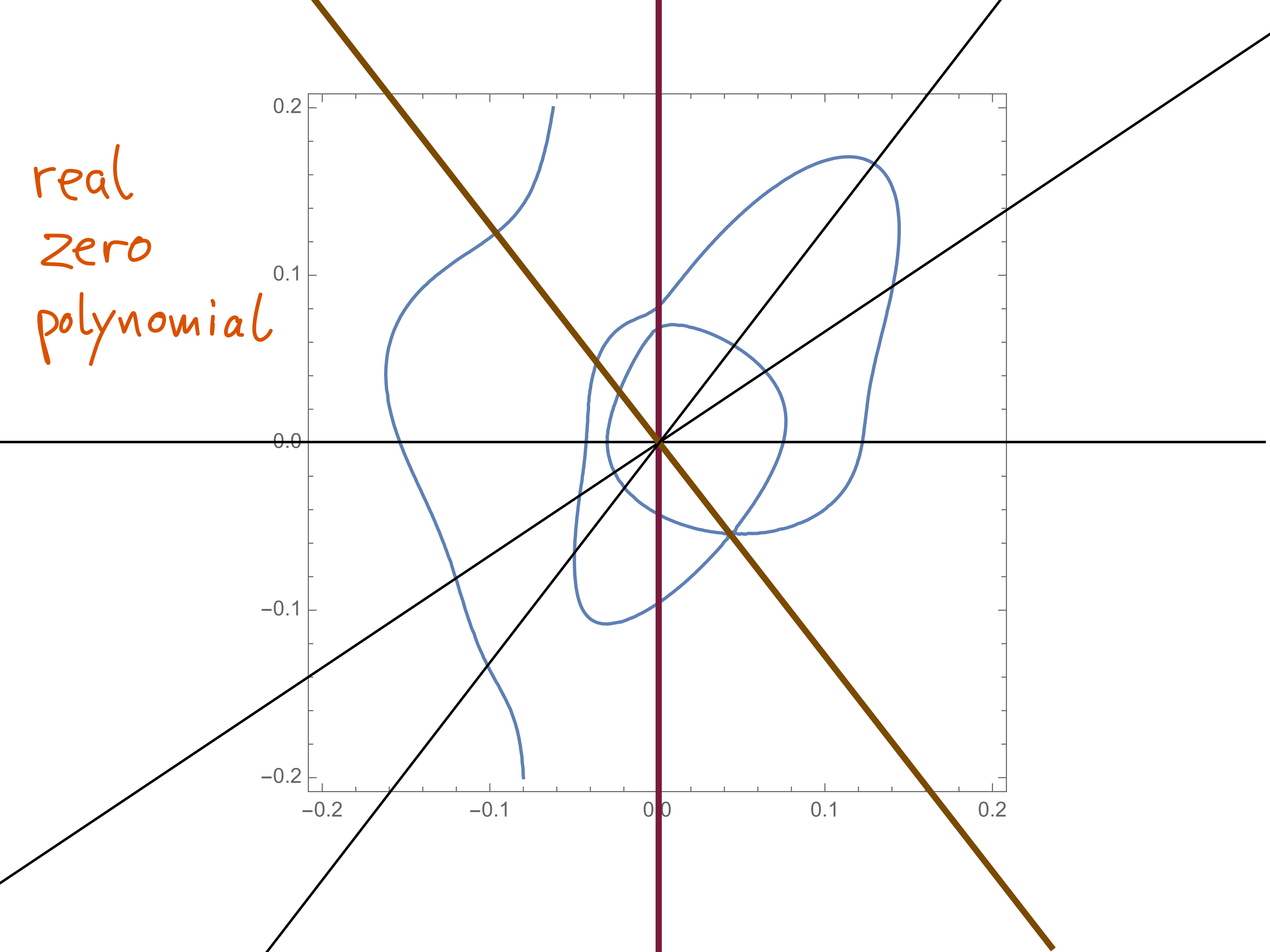
Very preliminary notes available:

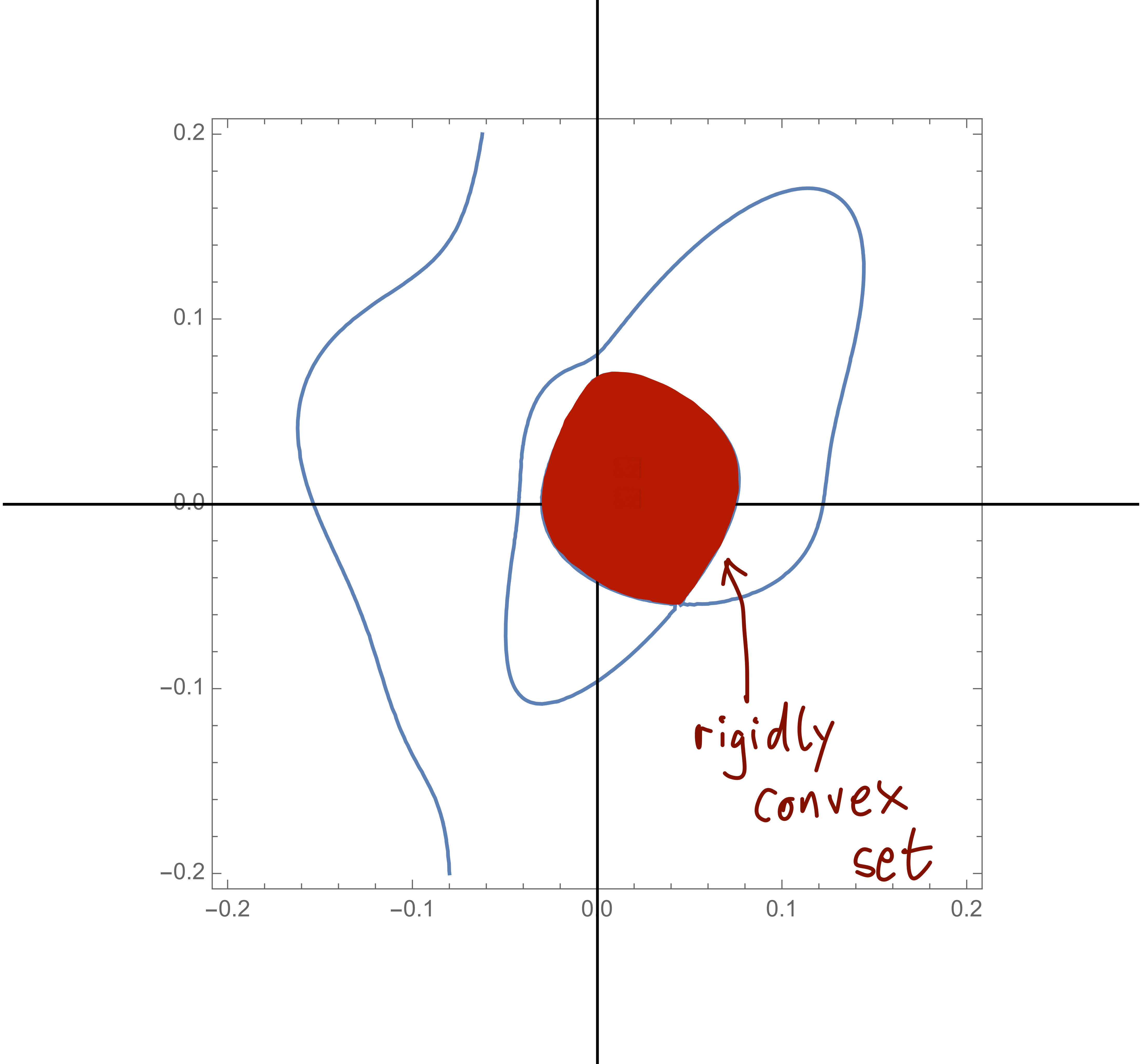
https://arxiv.org/abs/1907.13611

real zero set of $p = 1 + 42x_1 + 6x_2 + \dots - 37900\,x_2^5$



real
zero
polynomial




0.2
0.1
0.0
−0.1
−0.2
−0.2
−0.1
0.0
0.1
0.2
rigidly convex set

The generalized Lase conjecture (GLC)

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Helton & Vinnikov conjectured in 2007:

Every rigidly convex set is a spectrahedron.

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Impressive partial results by many people such as:

positive

- Helton & Vinnikov 2007
- Netzer & Thom 2013
- Brändén 2014
- Kummer 2016
- Saunderson 2018
- Amini 2019
- Kummer preprint 2020
- Vinnikov announced 2020

negative

- Raghavendra & Ryder & Srivastava & Weitz 2019
- Oliveira 2020

Our contribution to GLC

If RZAC or at least WRZAC holds, then we can "wrap each rigidly convex set into a spectrahedron and tie it with cords".



Our contribution to GLC

will explain in a minute

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If RZAC or at least WRZAC holds, then we can "wrap each rigidly convex set into a spectrahedron and tie it with cords".

will explain later

The real zero amalgamation conjecture (RZAC)

Conjecture: Suppose that $p \in \mathbb{R}[x,y]$ and $q \in \mathbb{R}[x,z]$ are real zero polynomials of degree at most d such that $p(x,0) = q(x,0)$.

ℓ variables m variables n variables

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Then there exists a real zero polynomial $r \in \mathbb{R}[x,y,z]$ of degree at most d such that $r(x,y,0) = p$ and $r(x,0,z) = q$.

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Can prove 3 special cases in 3 completely different ways:

$\ell = m = n = 1$	determinantal representations	Helton & Vinnikov
$\ell = 0$	stability preservers	Borcea & Brändén
$d = 2$	positive semidefinite matrix completion	Grone & Johnson & da Sa & Wolkowicz

The weak real zero amalgamation conjecture (WRZAC)

Conjecture: Suppose that $p \in \mathbb{R}[x,y]$ and $q \in \mathbb{R}[x,z]$ are real zero polynomials ~~of degree at most d~~ such that $p(x,0) = q(x,0)$.

ℓ variables m variables n variables

Then there exists a real zero polynomial $r \in \mathbb{R}[x,y,z]$ ~~of degree at most d~~ such that

$r(x,y,0) = p$ and $\operatorname{trunc}_3 r(x,0,z) = \operatorname{trunc}_3 q$.

The weak real zero amalgamation conjecture (WRZAC)

Conjecture: Suppose that $p \in \mathbb{R}[x,y]$ and $q \in \mathbb{R}[x,z]$ are real zero polynomials ~~of degree at most d~~ such that $p(x,0) = q(x,0)$.

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Then there exists a real zero polynomial $r \in \mathbb{R}[x,y,z]$ ~~of degree at most d~~ such that

$r(x,y,0) = p$ and $r(x,0,z) = q$.

trunc_3 trunc_3

This is trivial for $\ell = 0$.

Of course, RZAC $\Longrightarrow$ WRZAC.

Wrapping rigidly convex sets into spectrahedra and tying them with cords

Theorem: Suppose that RZAC or at least WRZAC holds. Then given a rigidly convex set and finitely many planes in $\mathbb{R}^n$, there is a spectrahedron in $\mathbb{R}^{n'}$ containing the rigidly convex set and agreeing with it on each of the planes.

Wrapping rigidly convex sets into spectrahedra and tying them with cords

Theorem: Suppose that RZAC or at least WRZAC holds. Then given a rigidly convex set and finitely many planes in $\mathbb{R}^n$, there is a spectrahedron in $\mathbb{R}^{n'}$ containing the rigidly convex set and agreeing with it on each of the planes.

Won't explain the proof but it uses again Helton & Vinnikov and a construction we now want to present.

Wrapping rigidly convex sets into spectrahedra and tying them with ribbons

Theorem: Suppose that RZAC or at least WRZAC holds. Then given a rigidly convex set (defined by a cubic real zero polynomial) and finitely many three-dimensional subspaces of $\mathbb{R}^n$, there is a spectrahedron in $\mathbb{R}^n$ containing the rigidly convex set and agreeing with it on each of these subspaces.

Won't explain the proof but it uses a result of Buckley & Košir and a construction we now want to present.

Exponential and logarithm on power series

$\mathbb{R}[[x]] := \mathbb{R}[[x_1, \dots, x_n]]$ ring of formal power series

$$\underset{B}{A} := \{ p \in \mathbb{R}[[x]] \mid p(0) = \underset{1}{0} \}$$

$$\exp : A \to B, \quad p \mapsto \sum_{k=0}^{\infty} \frac{p^k}{k!}$$

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$$\forall p, q \in A : \exp(\overbrace{p+q}^{\in A}) = (\exp p)(\exp q)$$

$$\forall p, q \in B : \log(\underbrace{pq}_{\in B}) = (\log p) + (\log q)$$

The associated linear form

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

The associated linear form

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Here always p a real zero polynomial and often $d = \deg p$.

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Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

$L_{p,d} : \mathbb{R}[x] \to \mathbb{R}$ linear

"linear form associated to p with respect to the virtual degree d"

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"linear form associated to p with respect to the virtual degree d"

$$L_{p,d}(1) = d$$

$$-\log \frac{p(-x)}{p(0)} = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ \alpha \neq 0}} \frac{1}{|\alpha|} \binom{|\alpha|}{\alpha} L_p(x^\alpha) x^\alpha$$

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"linear form associated to p"

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"linear form associated to p"

$$\forall p,q \in \mathbb{R}[x] : (p(0) \neq 0 \neq q(0) \Longrightarrow L_{pq} = L_p + L_q)$$

The associated linear form

If $d \in \mathbb{N}_0$, $a_1, \dots, a_d \in \mathbb{R}^n$ and $p := \prod_{i=1}^{d} (1 + a_i^T x)$, then $\forall q \in \mathbb{R}[x] : L_p(q) = \sum_{i=1}^{d} q(a_i)$.

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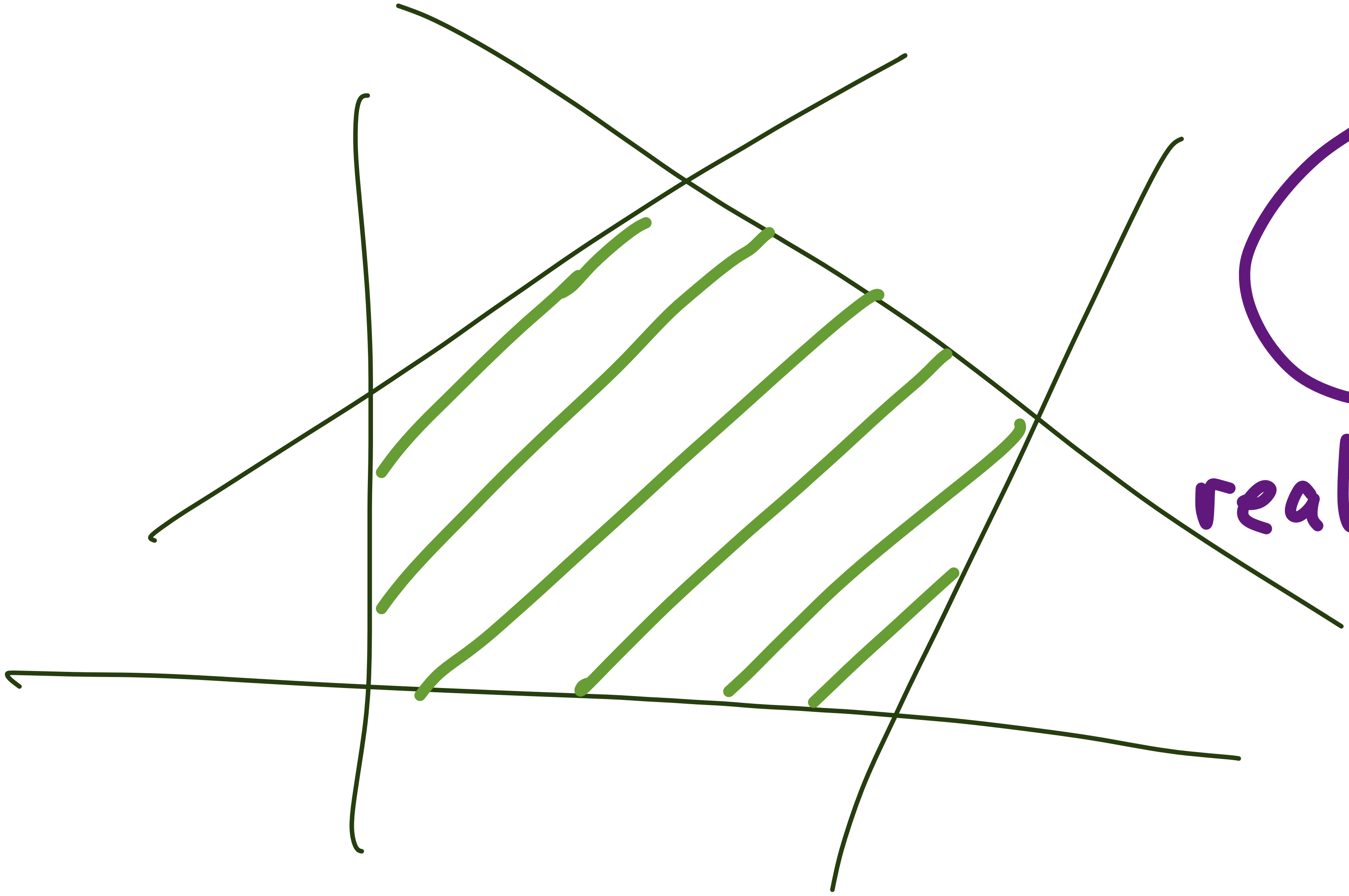
$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

prototype of all real zero polynomials



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If $d \in \mathbb{N}_0$ and $A_1, \ldots, A_n \in \mathbb{C}^{d \times d}$ are hermitian, then $p := \det(I_d + x_1 A_1 + \ldots + x_n A_n) \in \mathbb{R}[x]$ and $L_{p,d}(x^\alpha) = \operatorname{tr}(\operatorname{hur}_\alpha(A_1, \ldots, A_n))$ for all $\alpha \in \mathbb{N}_0^n$.

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"α-Hurwitz product"

$$\operatorname{hur}_{(2,3)}(A_1, A_2) = \frac{1}{\binom{5}{2\;3}} \Big(\underbrace{A_1 A_1 A_2 A_2 A_2 + A_1 A_2 A_1 A_2 A_2 + \dots}_{\binom{5}{2\;3} \text{ terms}} \Big)$$

The associated linear form

If $d \in \mathbb{N}_0$, $a_1, \dots, a_d \in \mathbb{R}^n$ and $p := \prod_{i=1}^{d} (1 + a_i^T x)$,
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and $L_{p,d}(x^\alpha) = \operatorname{tr}(\operatorname{hur}_\alpha(A_1, \dots, A_n))$ for all $\alpha \in \mathbb{N}_0^n$.

Trivial but crucial observation:

If $|\alpha| \leq 3$ and either $n \leq 2$
or ($n \leq 3$ and each A_i real),
the A_i commute "here".

The associated pencil

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

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Here always p a real zero polynomial and often $d = \deg p$.

The associated pencil

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

$M_{p,d} := A_0 + x_1 A_1 + \ldots + x_n A_n \in \mathbb{R}[x]^{(n+1)\times(n+1)}$

"pencil associated to p with respect to the virtual degree d"

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"pencil associated to p with respect to the virtual degree d"

$A_i \in \mathbb{R}^{(n+1)\times(n+1)}$ arises from $x_i \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} (x_0\; x_1 \ldots x_n)$

by substituting x_0 with 1 and then applying $L_{p,d}$ entrywise.

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If $p \in \mathbb{R}[x]$, set $M_p := M_{p, \deg p}$.

"linear form associated to p"

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Only degree 3

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If $p \in \mathbb{R}[x]$, set $M_p := M_{p,\deg p}$.

"linear form associated to p"

If p is a product of d linear forms, then A_0 is a moment matrix and the A_i are localization matrices.

The associated pencil

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$A_i \in \mathbb{R}^{(n+1)\times(n+1)}$ arises from $x_i \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} (x_0\; x_1 \ldots x_n)$ by substituting x_0 with 1 and then applying $L_{p,d}$ entrywise.

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If p is a product of d linear forms, then A_0 is a moment matrix and the A_i are localization matrices.

Only degree 3

then also higher degree

The associated spectrahedron

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

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The associated spectrahedron

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

$$S_d(p) := \{ a \in \mathbb{R}^n \mid M_{p,d}(a) \succeq 0 \}$$

"spectrahedron associated to p with respect to the virtual degree d"

The associated spectrahedron

Suppose $p \in \mathbb{R}[[x]]$, $p(0) \neq 0$ and $d \in \mathbb{N}_0$.

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"spectrahedron associated to p with respect to the virtual degree d"

If $p \in \mathbb{R}[x]$, set $S(p) := S_{\deg p}(p)$.

"spectrahedron associated to p"

The associated spectrahedron

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If p is a product of d linear forms, more precisely $d \in \mathbb{N}_0$, $a_1, \dots, a_d \in \mathbb{R}^n$ and $p = \prod_{i=1}^{d} (1 + a_i^T x)$, then $S(p) = \{a \in \mathbb{R}^n \mid \forall q \in \mathbb{R}[x]_1 : \sum_{i=1}^{d} q(a_i)^2 (1 + a_i^T x) \geq 0\}$

The associated spectrahedron

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COULD DO IT FOR HIGHER DEGREE $\rightsquigarrow$ finitely converging hierarchy

A key lemma

Let $d \in \mathbb{N}_0$, set $V := \{M \in \mathbb{R}^{d \times d} \mid M \text{ symmetric}\}$ and let $A_1, \dots, A_n \in V$. Then

$$p := \det(I_d + x_1 A_1 + \dots + x_n A_n) \in \mathbb{R}[x]$$

is a real zero polynomial with

$$C(p) = \{a \in \mathbb{R}^n \mid I_d + a_1 A_1 + \dots + a_n A_n \succeq 0\}$$

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and $S_d(p) = \{a \in \mathbb{R}^n \mid \forall \in \underset{\sim}{U} : \operatorname{tr}(M^2(I_d + a_1 A_1 + \dots + a_n A_n)) \geq 0\}$

where $U := \{\lambda_0 I_d + \lambda_1 A_1 + \dots + \lambda_n A_n \mid \lambda_0, \lambda_1, \dots, \lambda_n \in \mathbb{R}\} \subseteq V.$

A key lemma

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possibly $\deg p < d$

is a real zero polynomial with

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where $U := \{\lambda_0 I_d + \lambda_1 A_1 + \dots + \lambda_n A_n \mid \lambda_0, \lambda_1, \dots, \lambda_n \in \mathbb{R}\} \subseteq V.$

A key lemma

Let $d \in \mathbb{N}_0$, set $V := \{M \in \mathbb{R}^{d\times d} \mid M \text{ symmetric}\}$ and let $A_1,\dots,A_n \in V$. Then

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If $n \leq 2$, then $= \{ a \in \mathbb{R}^n \mid \forall M \in V : \operatorname{tr}(M^2 (I_d + a_1 A_1 + \dots + a_n A_n)) \geq 0 \}$

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If $p \in \mathbb{R}[x_1, x_2]$ is a real zero polynomial of degree d with $p(0) = 1$, then there exist symmetric $A_1, A_2 \in \mathbb{R}^{d \times d}$ such that

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Two variables only!

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weaker version

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Proof ~~very hard~~ easier: Helton & Vinnikov 2006
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e.g. Grinshpan & Kaliuzhnyi-Verbovetskyi & Vinnikov & Woerdeman 2016

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affine version

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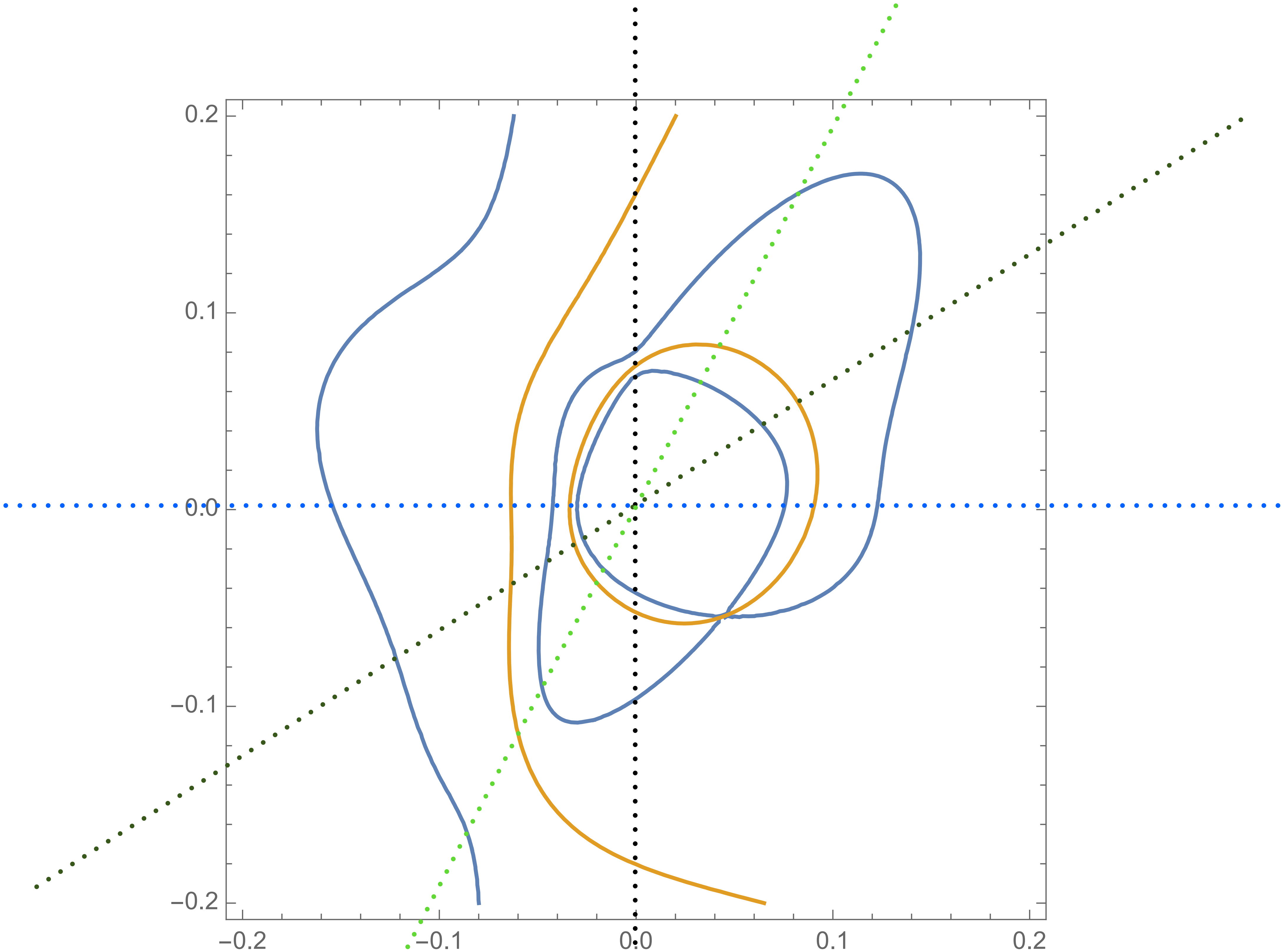
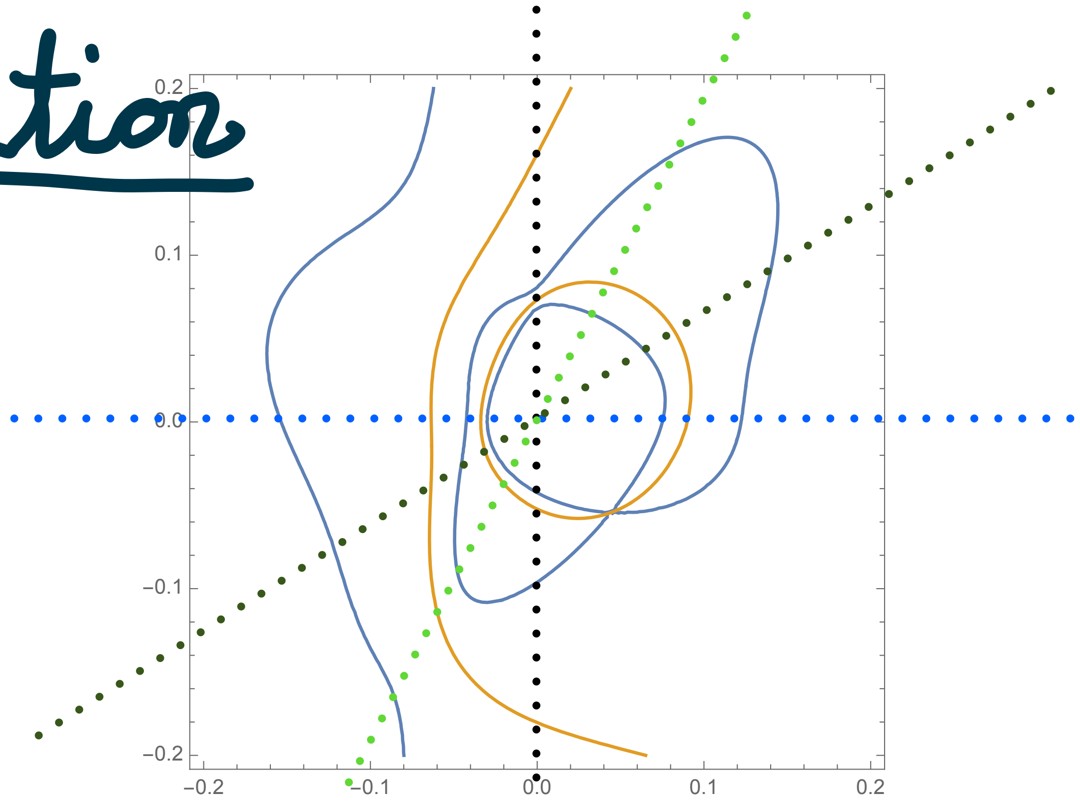
A bit trickier than it seems:

- need compatibility of construction with orthogonal maps
- degree could drop.

Interlacers



Tightening the relaxation



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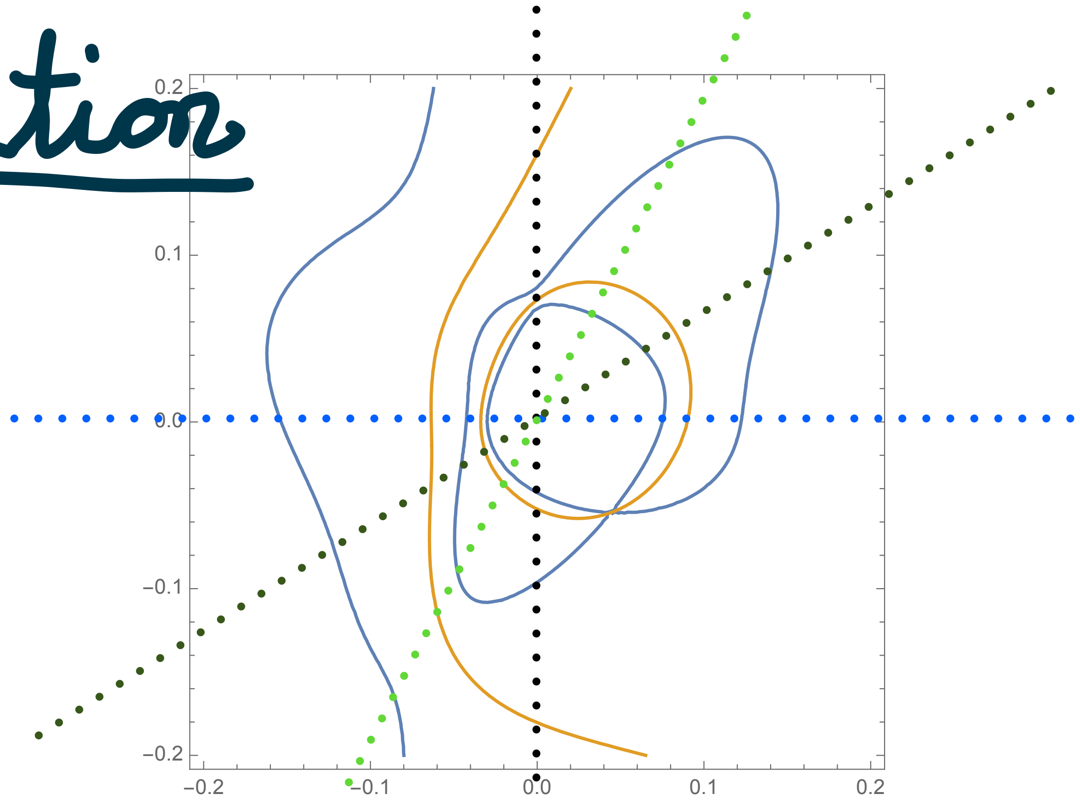

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Idea: $S_d(p) \rightsquigarrow \{a \in \mathbb{R}^n \mid (a,0) \in S_d(p+uq)\}$

Two hardness results

first one:

Raghavendra & Ryder & Srivastava & Weitz (2019)

$\exists N \in \mathbb{N} : \forall n, d \geq N$: there is a real zero polynomial in n variables of degree d whose rigidly convex set is not defined by a linear matrix inequality of size at most $\left(\frac{n}{d}\right)^{\frac{d}{N}}$.

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exponential growth for, e.g., $n = 2d$:

$$\left(\frac{n}{d}\right)^{\frac{d}{N}} \geq 2^{\frac{d}{N}} \geq 2^{\frac{1}{N}}$$

Two hardness results

second one:

Oliveira (2020) under the hypothesis $VP \neq VNP$ gives explicit example for superpolynomial growth based on Amini's multivariate matching polynomials of complete bipartite graphs.

Attacking GLC?

If p is a real zero polynomial in n variables of degree d, $S_d(p)$ depends only on $\mathrm{trunc}_3\, p$ and d and is defined by a linear matrix inequality of size $d+1$.

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