Pure states and the stability number of graphs

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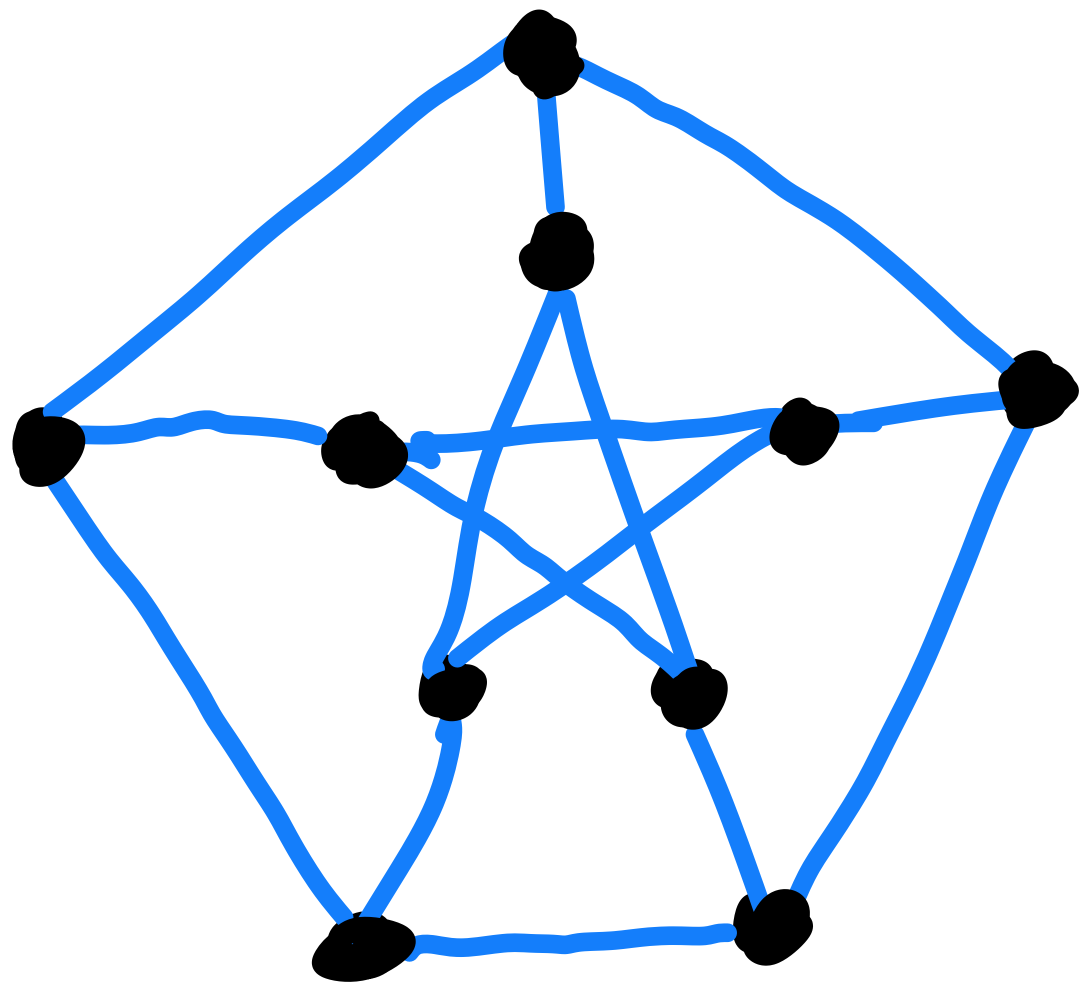
$$\alpha(G) := \max\{\#S \mid S \text{ stable set of } G\}$$

("maximum cardinality of a stable set").

Ex 2



Peterson graph

10 nodes

15 edges

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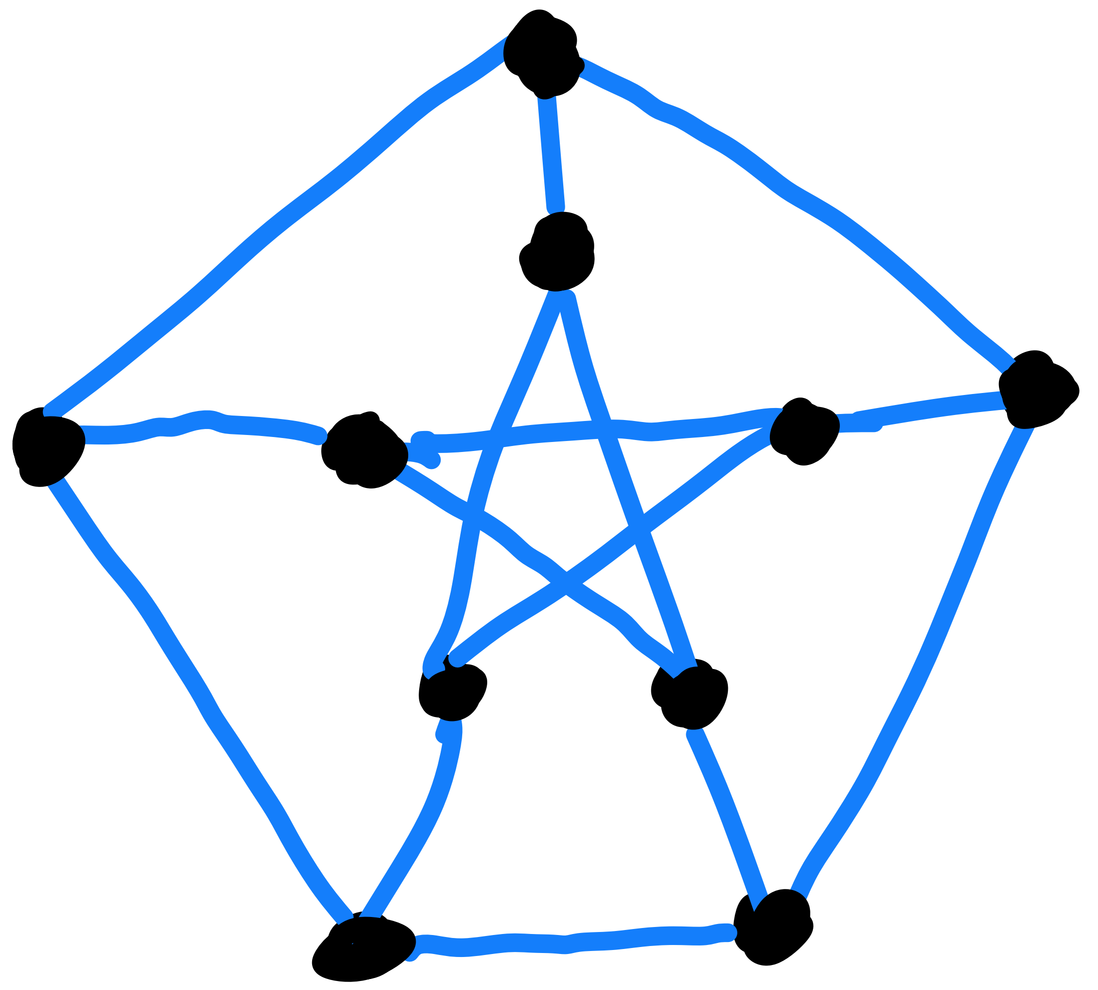

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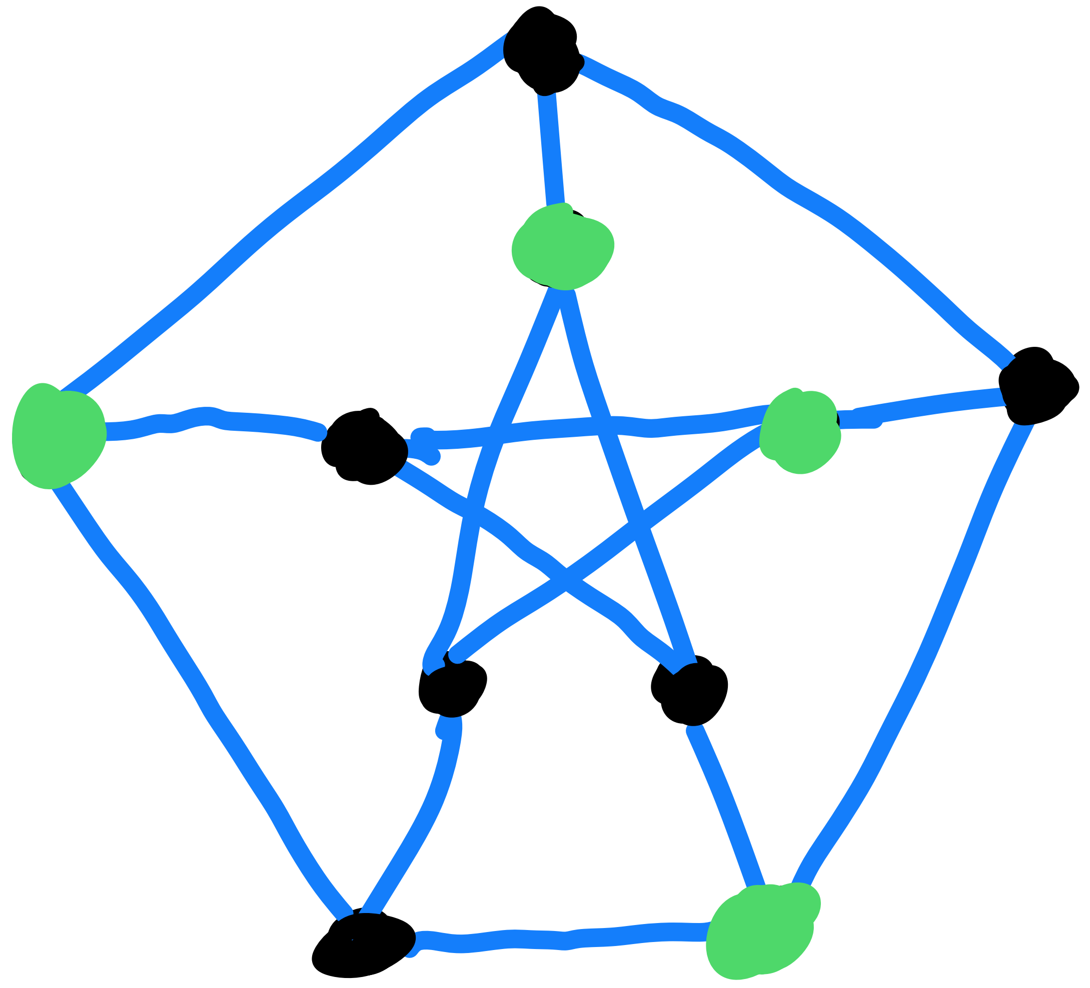

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$$f_G := \alpha(G)\left(\sum_{\substack{i,j\in V\\ i=j \text{ or } \{i,j\}\in E}} X_i^2 X_j^2\right) - \left(\sum_{i\in V} X_i^2\right)^2 \in \mathbb{R}[X_i \mid i\in V]$$

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Rem 5 If S is a stable set of the graph $G=(V,E)$ with $\#S=\alpha(G)$, then $f_G(\mathbb{1}_S)=0$.

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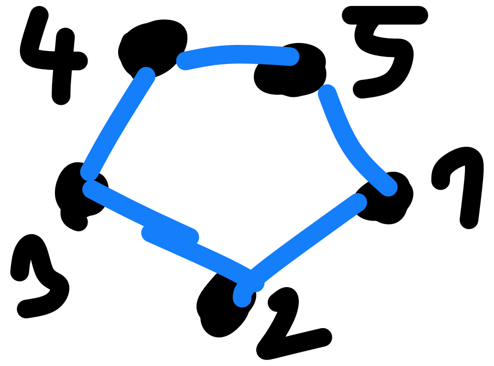
Ex 8 If $G =$ 

("five cycle"), then $f_G \in \mathbb{R}[X_1, X_2, X_3, X_4, X_5]$ is the Horn form which Choi and Lam knew to be not sos and for which Parrilo showed $(X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2) f_G$ to be sos.

Conj 9 (de Klerk & Pasechnik, 2002)

If $G=(V,E)$ is a graph with $V \neq \emptyset$, then $\left(\sum_{i\in V} x_i^2\right)^{\alpha(G)-1} f_G$ is sos.

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In the remaining, we try to briefly sketch a proof of this result. The proof relies on graph theory and real algebraic geometry. We focus on the latter.

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<u>Lem 11</u> Let $G=(V,E)$ be a graph, let $0 \notin V$ and set $H := (V \cup \{0\}, E)$ ("add one isolated node").

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then there is some $s \in \mathbb{N}_0$ such that

$$\left(X_0^2+\sum_{i\in V} X_i^2\right)^s f_H \text{ is sos.}$$

Lem 12 (de Klerk & Laurent & Parrilo 2005)

Let $f \in \mathbb{R}[X_1, \dots, X_n]$ be homogeneous of even degree.

Set $M := \left\{ \sigma + p \cdot \left(1 - \sum_{i=1}^{n} X_i^2\right) \mid \sigma \text{ sos}, \, p \in \mathbb{R}[X_1, \dots, X_n] \right\}$.

Then $f \in M$ if and only if there is some $s \in \mathbb{N}_0$ such that $\left(\sum_{i=1}^{n} X_i^2\right)^s f$ is sos.

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Rem 13 M from Lemma 12 is an example of an "Archimedean quadratic module". If you don't know what this is, then take always this M in the sequel.

Thm 14 Let M be an Archimedean quadratic module of $\mathbb{R}[X_1,\dots,X_n]$, $S:=\{x\in\mathbb{R}^n \mid \forall p\in M: p(x)\geq 0\}$, $f\in\mathbb{R}[X_1,\dots,X_n]$ with $f\geq 0$ on S and $Z:=\{x\in S \mid f(x)=0\}$.

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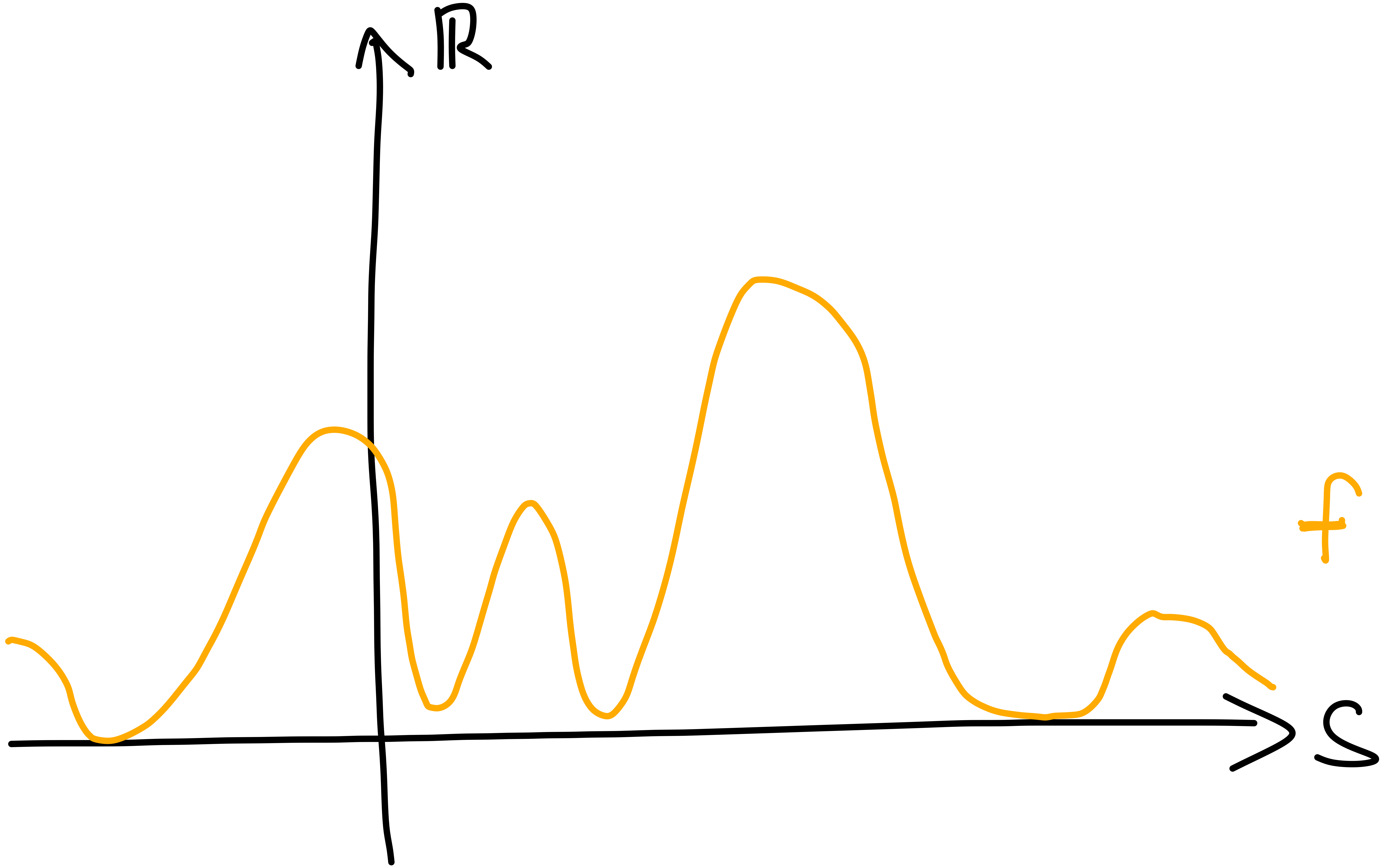
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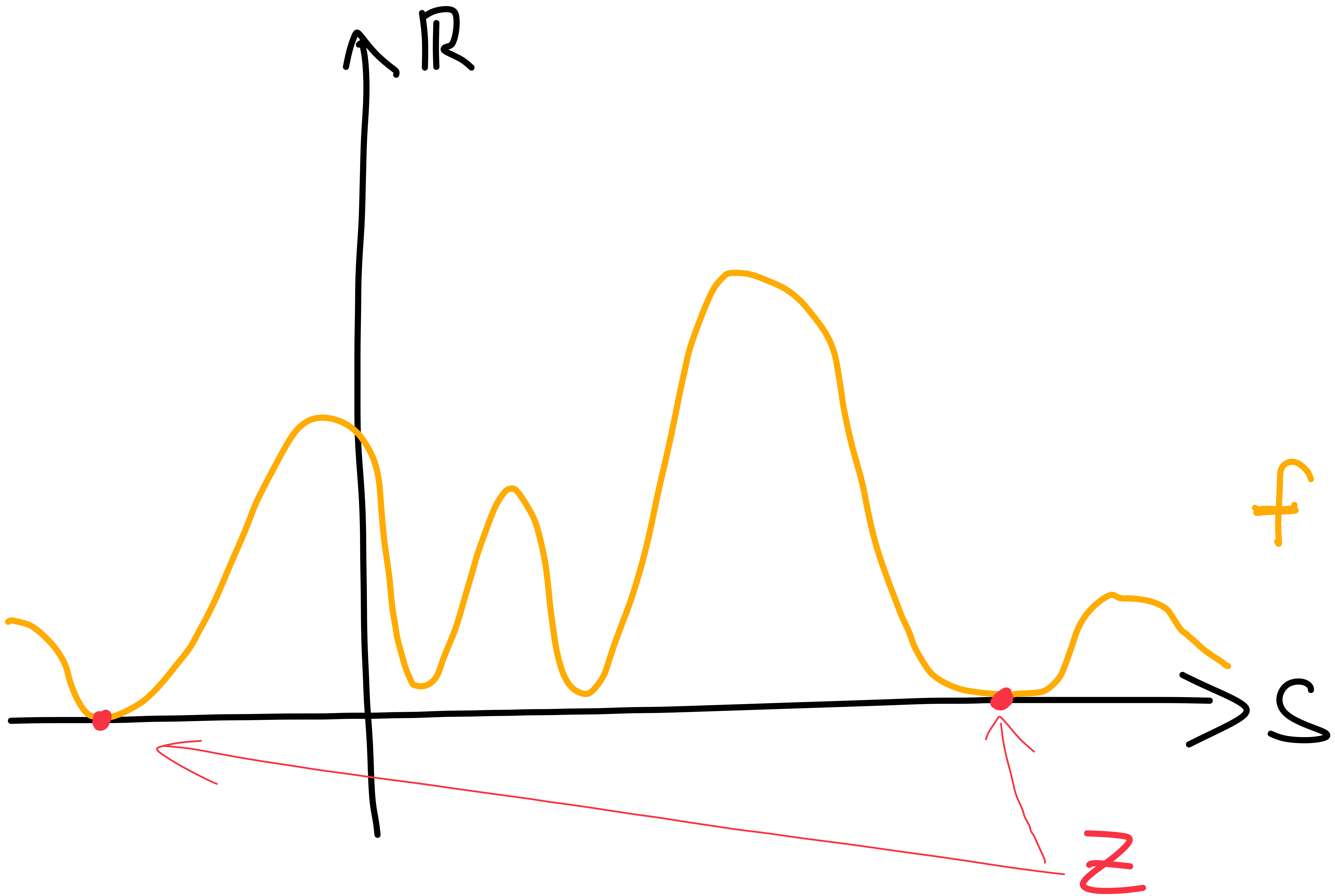

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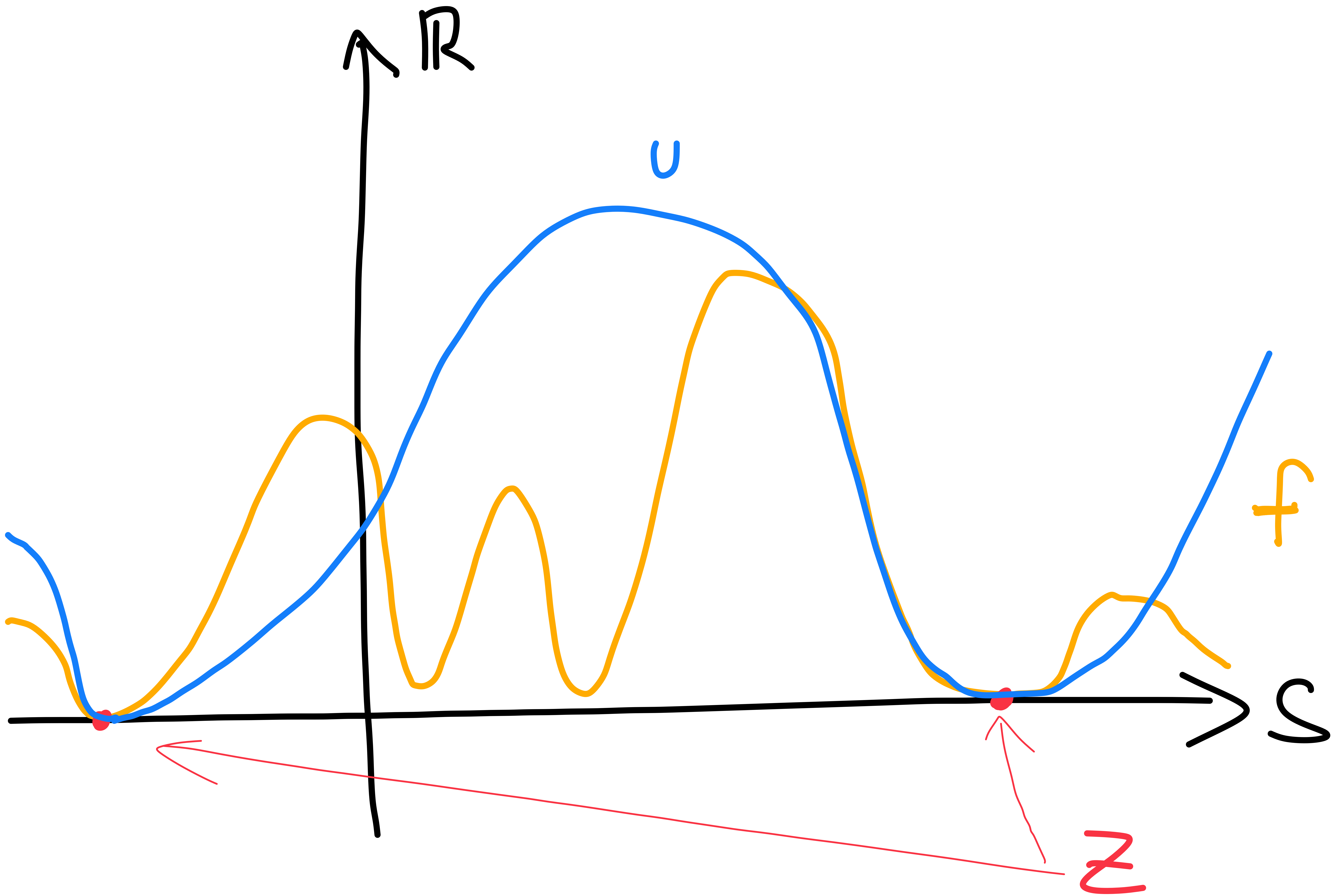

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The proof of Theorem 14 is based on the theory of pure states on ideals (Burgdorf & Scheiderer & S., 2012).

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Also visit my lecture notes on Real Algebraic Geometry, Positivity and Convexity

arxiv.org/abs/2205.04211

and the corresponding YouTube playlist:

https://youtube.com/playlist?list=PLbQ93L5pV-a_RRwdEgGungHn5rN43BGe7

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$$f_H \in M := \left\{ \sigma + p \cdot \left(1 - \sum_{i=1}^{n} X_i^2\right) \;\middle|\; \sigma \text{ sos},\ p \in \mathbb{R}[X_1, \dots, X_n] \right\}.$$

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Set $S := \{x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + x_1^2 + \dots + x_n^2 = 1\}$

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Observe that $f_H = g^2 + \frac{\alpha(G)+1}{\alpha(G)} f_G$ where

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Set $G := \{g^2, f_G\}$ and let I be the ideal generated by G.

Of course, $f_H \in I$. Set $u := g^2 + \frac{\alpha(G)+1}{\alpha(G)} \left(\sum_{i=1}^{n} X_i^2\right)^{4r} f_G$.

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We have a tricky proof showing that for all $g \in G$ there is an $\varepsilon > 0$ such that $v \pm \varepsilon g \in \mathcal{M}$.

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It is clear that $v \in I$, and since v is sos, it is trivial that $vM \subseteq M$.

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By Theorem 6 of Motzkin & Straus, we know that $f_H \geq 0$ on S.

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Now suppose that $x \in Z$ and let $\varphi : I \to \mathbb{R}$ be linear such that $\varphi(M \cap I) \subseteq \mathbb{R}_{\geq 0}$, $\varphi(u) = 1$ and

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By Theorem 14, it suffices to show $\varphi(f_H) > 0$.

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$\varphi(f_H) = \underbrace{\varphi(g^2)}_{\geq 0} + \underbrace{\frac{\alpha(G)+1}{\alpha(G)}}_{>0} \underbrace{\varphi(f_G)}_{\geq 0} > 0$ for otherwise $\underbrace{\varphi(u) = \varphi(g^2) + \dots \varphi(f_G)}_{=0}$