On the exactness of moment relaxations

(joint work with Tom-Lukas Kriel)

Markus Schweighofer

Universität Konstanz

Dagstuhl seminar on limitations of convex programming
February 15-20, 2015

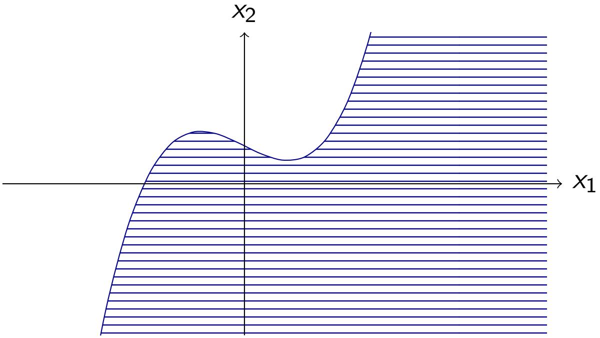
System of polynomial inequalities

$$
\begin{array}{rrrrrrrrrrrr}
 & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
- & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
 & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

System of polynomial inequalities

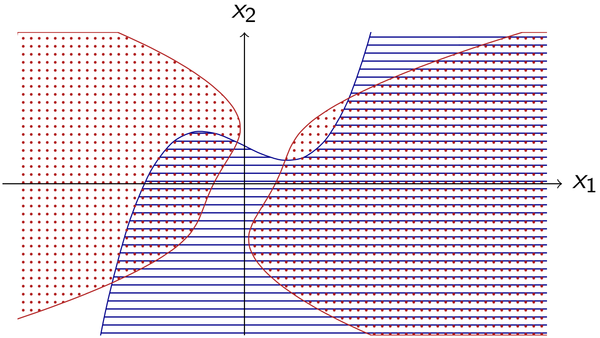
A

$$\begin{array}{rrrrrrrl} & & x_1^3 & - & x_1 & - & 2x_2 & + 1 \geq 0 \\ - x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - \frac{1}{3} \geq 0 \\ & - & x_1^2 & - & x_2^2 & + & x_1 & + 4 \geq 0 \end{array}$$



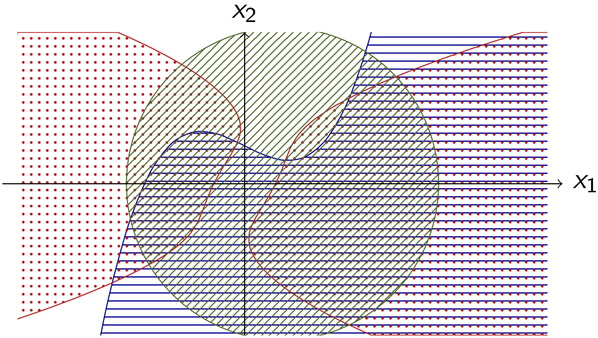
System of polynomial inequalities

$$\begin{array}{lrcl}
A & x_1^3 - x_1 - 2x_2 + 1 & \geq & 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq & 0 \\
 & -x_1^2 - x_2^2 + x_1 + 4 & \geq & 0
\end{array}$$



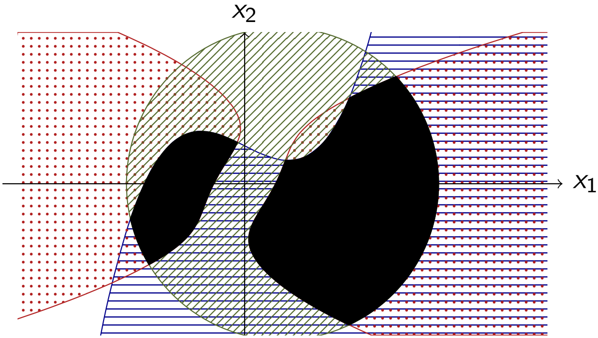
System of polynomial inequalities

A					x_1^3	$-$	x_1	$-$	$2x_2$	$+$	1	$\geq$	0
B	$-$	x_2^4	$+$		$2x_1^2$	$-$	$2x_1x_2$	$+$	x_2^2	$-$	$\frac{1}{3}$	$\geq$	0
C			$-$		x_1^2	$-$	x_2^2	$+$	x_1	$+$	4	$\geq$	0



System of polynomial inequalities

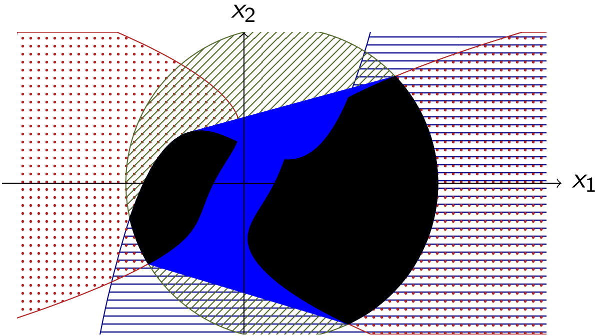
A	$x_1^3 - x_1 - 2x_2 + 1$	≥ 0
B	$-x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3}$	≥ 0
C	$-x_1^2 - x_2^2 + x_1 + 4$	≥ 0



S

System of polynomial inequalities

A					x_1^3	$-$	x_1	$-$	$2x_2$	$+$	1	$\geq$	0
B	$-$	x_2^4	$+$	$2x_1^2$	$-$	$2x_1x_2$	$+$	x_2^2	$-$	$\frac{1}{3}$	$\geq$	0	
C			$-$	x_1^2	$-$	x_2^2	$+$	x_1	$+$	4	$\geq$	0	



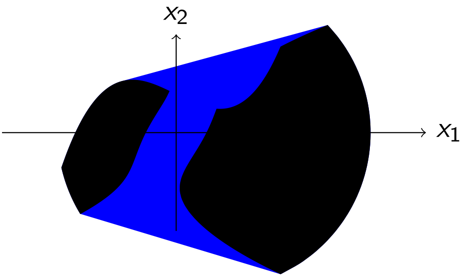
S

conv S

System of polynomial inequalities

Very naive linearization

$$
\begin{array}{llr}
A & x_1^3 - x_1 - 2x_2 + 1 & \geq 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 & \geq 0
\end{array}
$$



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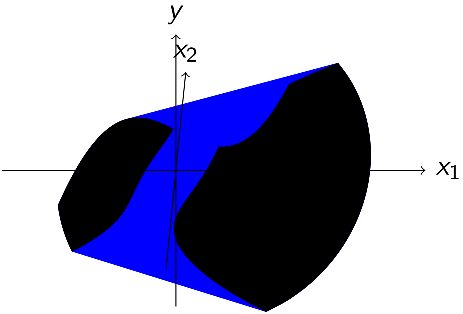
System of polynomial inequalities

Very naive linearization

$$
\begin{array}{lrcl}
A & x_1^3 - x_1 - 2x_2 + 1 & \geq & 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq & 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 & \geq & 0
\end{array}
$$



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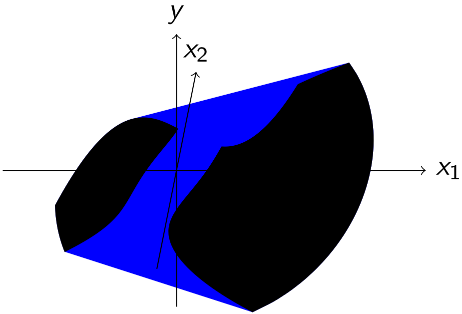
System of polynomial inequalities

Very naive linearization

$$\begin{array}{lrcl}
A & x_1^3 - x_1 - 2x_2 + 1 & \geq & 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq & 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 & \geq & 0
\end{array}$$



S

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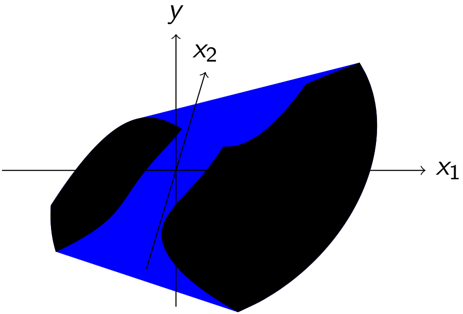
System of polynomial inequalities

Very naive linearization

A	$x_1^3 - x_1 - 2x_2 + 1 \geq 0$
B	$-x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} \geq 0$
C	$-x_1^2 - x_2^2 + x_1 + 4 \geq 0$



S

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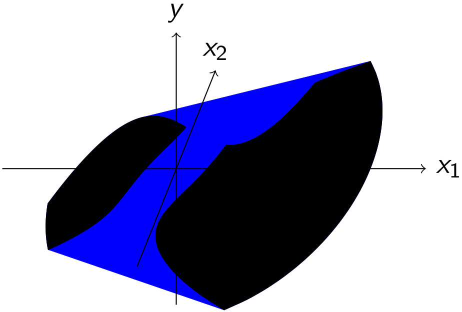
System of polynomial inequalities

Very naive linearization

$$
\begin{array}{lrrrrrl}
A & & x_1^3 & - x_1 & - 2x_2 & + 1 & \geq 0 \\
B & - x_2^4 & + 2x_1^2 & - 2x_1x_2 & + x_2^2 & - \frac{1}{3} & \geq 0 \\
C & & - x_1^2 & - x_2^2 & + x_1 & + 4 & \geq 0
\end{array}
$$



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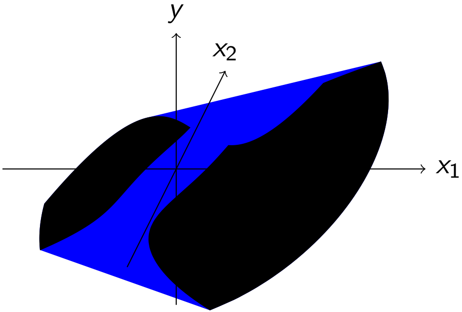
System of polynomial inequalities

Very naive linearization

$$
\begin{array}{lrrrrrl}
A & & x_1^3 & - x_1 & - 2x_2 & + 1 & \geq 0 \\
B & - x_2^4 & + 2x_1^2 & - 2x_1x_2 & + x_2^2 & - \frac{1}{3} & \geq 0 \\
C & & - x_1^2 & - x_2^2 & + x_1 & + 4 & \geq 0
\end{array}
$$



S

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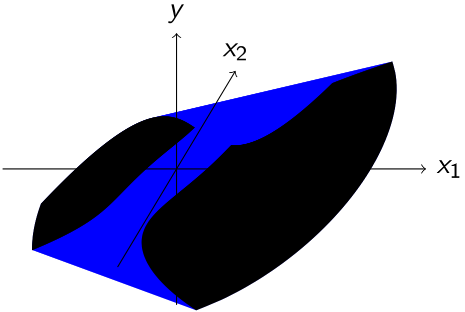
System of polynomial inequalities

Very naive linearization

$$
\begin{array}{lrcl}
A & x_1^3 - x_1 - 2x_2 + 1 & \geq & 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq & 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 & \geq & 0
\end{array}
$$



S

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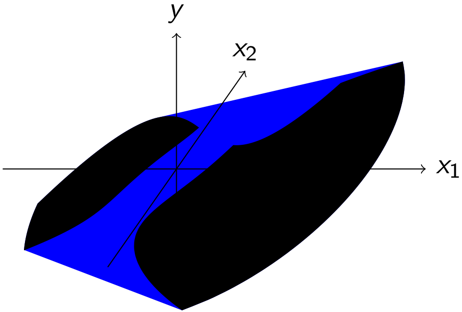
System of polynomial inequalities

Very naive linearization

$$
\begin{array}{lrrrrrrr}
A & & x_1^3 & - x_1 & - 2x_2 & + 1 & \geq & 0 \\
B & - x_2^4 & + 2x_1^2 & - 2x_1x_2 & + x_2^2 & - \frac{1}{3} & \geq & 0 \\
C & & - x_1^2 & - x_2^2 & + x_1 & + 4 & \geq & 0
\end{array}
$$



S

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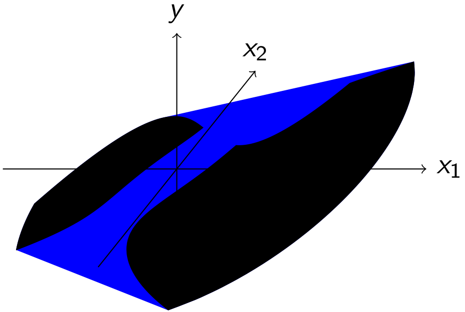
System of polynomial inequalities

Very naive linearization

A					x_1^3	$-$	x_1	$-$	$2x_2$	$+$	1	$\geq$	0
B	$-$	x_2^4	$+$	$2x_1^2$		$-$	$2x_1x_2$	$+$	x_2^2	$-$	$\frac{1}{3}$	$\geq$	0
C				$-$	x_1^2	$-$	x_2^2	$+$	x_1	$+$	4	$\geq$	0



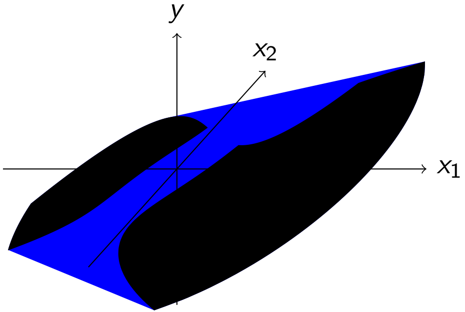
S

conv S

System of polynomial inequalities

Very naive linearization

$$\begin{array}{lrrrrrrl}
A & & x_1^3 & - x_1 & - 2x_2 & + 1 & \geq 0 \\
B & - x_2^4 + 2x_1^2 & - 2x_1x_2 & + x_2^2 & - \frac{1}{3} & & \geq 0 \\
C & & - x_1^2 & - x_2^2 & + x_1 & + 4 & \geq 0
\end{array}$$



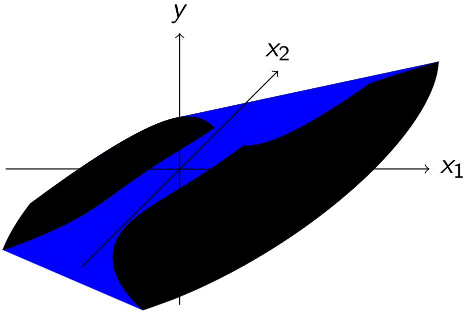
conv S

S

System of polynomial inequalities

Very naive linearization

$$\begin{array}{lrrrrrrrrrrrr}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

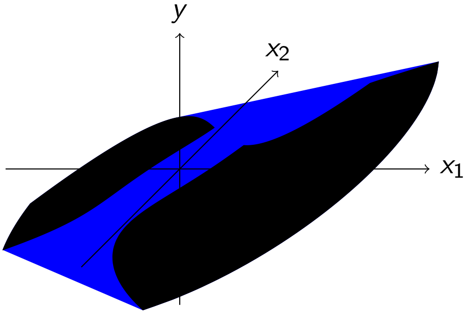


conv S

System of polynomial inequalities

Very naive linearization

$$
\begin{array}{llcl}
A & & y_1 - x_1 - 2x_2 + 1 & \geq 0 \\
B & - x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & & \geq 0 \\
C & - x_1^2 - x_2^2 + x_1 + 4 & & \geq 0
\end{array}
$$

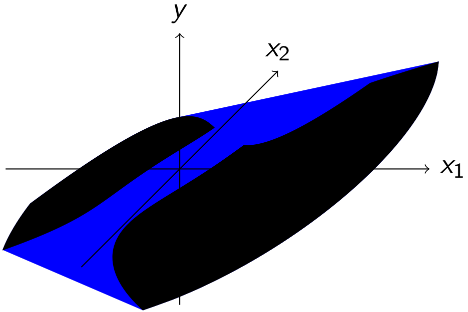


conv S

System of polynomial inequalities

Very naive linearization

A					y_1	$-$	x_1	$-$	$2x_2$	$+$	1	$\geq$	0
B	$-$	x_2^4	$+$	$2x_1^2$		$-$	$2x_1x_2$	$+$	x_2^2	$-$	$\frac{1}{3}$	$\geq$	0
C			$-$	x_1^2		$-$	x_2^2	$+$	x_1	$+$	4	$\geq$	0

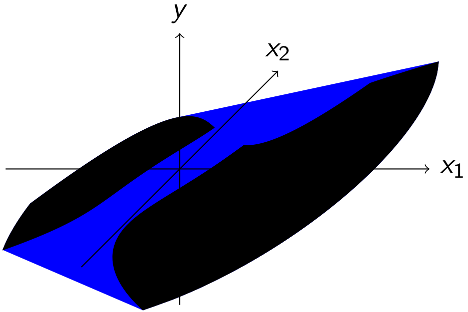


conv S

System of polynomial inequalities

Very naive linearization

$$
\begin{array}{lrcl}
A & y_1 - x_1 - 2x_2 + 1 & \geq & 0 \\
B & -y_2 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq & 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 & \geq & 0
\end{array}
$$

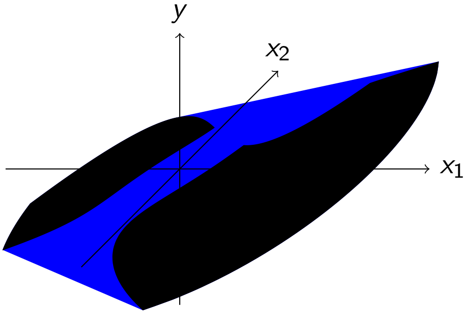


conv S

System of polynomial inequalities

Very naive linearization

$$
\begin{array}{lrl}
A & y_1 - x_1 - 2x_2 + 1 & \geq 0 \\
B & -y_2 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 & \geq 0
\end{array}
$$

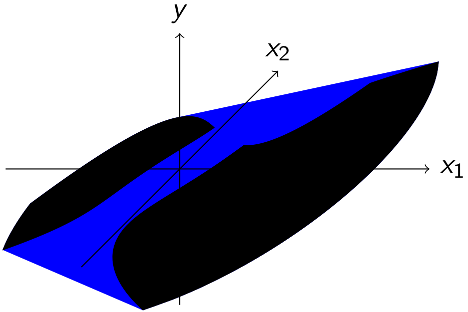


conv S

System of polynomial inequalities

Very naive linearization

$$\begin{array}{lrcl}
A & y_1 - x_1 - 2x_2 + 1 & \geq & 0 \\
B & -y_2 + 2y_3 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq & 0 \\
C & -y_3 - x_2^2 + x_1 + 4 & \geq & 0
\end{array}$$

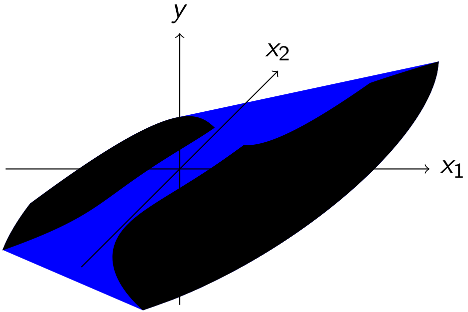


conv S

System of polynomial inequalities

Very naive linearization

$$
\begin{array}{lcrcrcrcrcrcl}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

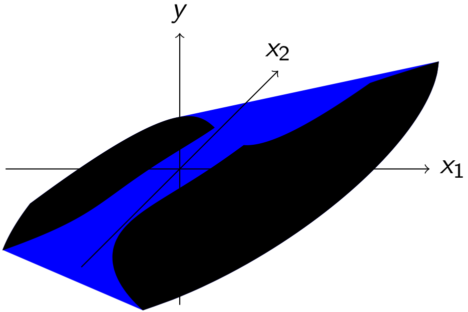


conv S

System of polynomial inequalities

Very naive linearization

$$
\begin{array}{lrrrrrrrrrrrr}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

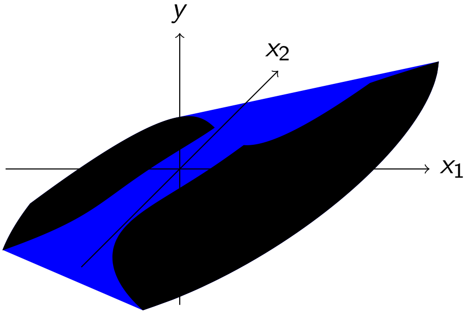


conv S

System of polynomial inequalities

Very naive linearization

$$
\begin{array}{lr}
A & y_1 - x_1 - 2x_2 + 1 \geq 0 \\
B & -y_2 + 2y_3 - 2y_4 + x_2^2 - \frac{1}{3} \geq 0 \\
C & -y_3 - x_2^2 + x_1 + 4 \geq 0
\end{array}
$$

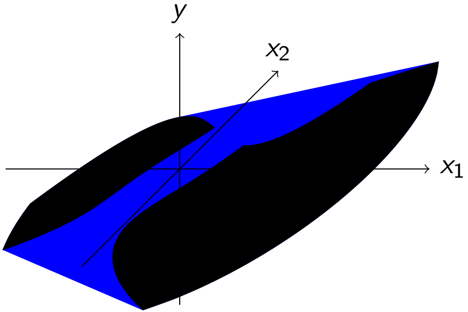


conv S

System of linear inequalities

Very naive linearization

$$\begin{array}{lrcl} A & y_1 - x_1 - 2x_2 + 1 & \geq & 0 \\ B & -y_2 + 2y_3 - 2y_4 + y_5 - \frac{1}{3} & \geq & 0 \\ C & -y_3 - y_5 + x_1 + 4 & \geq & 0 \end{array}$$

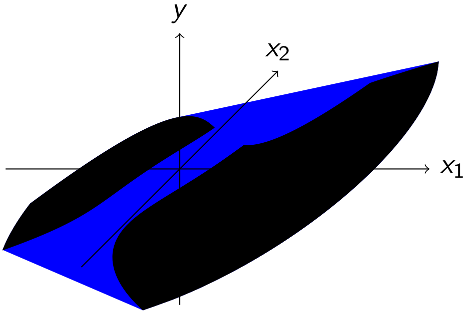


conv S

System of linear inequalities

Very naive linearization

$$\begin{array}{llr}
A & & y_1 - x_1 - 2x_2 + 1 \geq 0 \\
B & & -y_2 + 2y_3 - 2y_4 + y_5 - \frac{1}{3} \geq 0 \\
C & & -y_3 - y_5 + x_1 + 4 \geq 0
\end{array}$$



conv S

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrl}
A & & x_1^3 & - x_1 & - 2x_2 & + 1 & \geq 0 \\
B & - x_2^4 & + 2x_1^2 & - 2x_1x_2 & + x_2^2 & - \frac{1}{3} & \geq 0 \\
C & & - x_1^2 & - x_2^2 & + x_1 & + 4 & \geq 0
\end{array}
$$

redundant:

System of polynomial inequalities

Less naive linearization

$$\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} \\
AB & - & x_1^3x_2^4 & + & \ldots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0
\end{array}$$

System of polynomial inequalities

Less naive linearization

$$\begin{array}{lrrrrrrrrrrrrr}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} & & & & & & & & & & & & \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0
\end{array}$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} \\
AB & - & x_1^3x_2^4 & + & \ldots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \ldots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{llrcrcrcrcrcl}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} & & & & & & & & & & & & \\
AB & - & x_1^3x_2^4 & + & \ldots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \ldots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\multicolumn{14}{l}{\text{redundant:}} \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

A				x_1^3	$-$	x_1	$-$	$2x_2$	$+$	1	$\geq$	0
B	$-$	x_2^4	$+$	$2x_1^2$	$-$	$2x_1x_2$	$+$	x_2^2	$-$	$\frac{1}{3}$	$\geq$	0
C			$-$	x_1^2	$-$	x_2^2	$+$	x_1	$+$	4	$\geq$	0
redundant:												
AB	$-$	$x_1^3x_2^4$	$+$	$\dots$	$+$	x_2^2	$+$	$\frac{2}{3}x_2$	$-$	$\frac{1}{3}$	$\geq$	0
AC		x_1^5	$+$	$\dots$	$-$	x_1	$+$	$8x_2$	$-$	4	$\geq$	0
ABC	$-$	$x_1^5x_2^4$	$+$	$\dots$	$-$	$\frac{13}{3}x_2^2$	$-$	$\frac{8}{3}x_2$	$+$	$\frac{4}{3}$	$\geq$	0
D^2						x_1^2	$-$	$2x_1x_2$	$+$	x_2^2	$\geq$	0
D^2C	$-$	x_1^4	$+$	$\dots$	$+$	$4x_1^2$	$+$	$4x_1x_2$	$+$	$4x_2^2$	$\geq$	0

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrr}
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{llllllllllllll}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} & & & & & & & & & & & & \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrr}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrrr}
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

A					y_1	$-$	x_1	$-$	$2x_2$	$+$	1	$\geq$	0
B	$-$	y_2	$+$	$2y_3$		$-$	$2y_4$	$+$	x_2^2	$-$	$\frac{1}{3}$	$\geq$	0
C			$-$	y_3		$-$	x_2^2	$+$	x_1	$+$	4	$\geq$	0
irredundant:													
AB	$-$	$x_1^3x_2^4$	$+$	$\dots$		$+$	x_2^2	$+$	$\frac{2}{3}x_2$	$-$	$\frac{1}{3}$	$\geq$	0
AC		x_1^5	$+$	$\dots$		$-$	x_1	$+$	$8x_2$	$-$	4	$\geq$	0
ABC	$-$	$x_1^5x_2^4$	$+$	$\dots$		$-$	$\frac{13}{3}x_2^2$	$-$	$\frac{8}{3}x_2$	$+$	$\frac{4}{3}$	$\geq$	0
D^2							y_3	$-$	$2y_4$	$+$	x_2^2	$\geq$	0
D^2C	$-$	x_1^4	$+$	$\dots$		$+$	$4y_3$	$+$	$4y_4$	$+$	$4x_2^2$	$\geq$	0

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrr}
AB & - & x_1^3 x_2^4 & + & \ldots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \ldots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \ldots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{llllllllllllll}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{llllllllllllll}
AB & - & x_1^3 x_2^4 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2 C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrrr}
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

A					y_1	$-$	x_1	$-$	$2x_2$	$+$	1	$\geq$	0
B	$-$	y_2	$+$	$2y_3$	$-$	$2y_4$	$+$	y_5	$-$	$\frac{1}{3}$	$\geq$	0
C			$-$	y_3	$-$	y_5	$+$	x_1	$+$	4	$\geq$	0
irredundant:												
AB	$-$	y_6	$+$	$\dots$	$+$	y_5	$+$	$\frac{2}{3}x_2$	$-$	$\frac{1}{3}$	$\geq$	0
AC		x_1^5	$+$	$\dots$	$-$	x_1	$+$	$8x_2$	$-$	4	$\geq$	0
ABC	$-$	$x_1^5 x_2^4$	$+$	$\dots$	$-$	$\frac{13}{3}y_5$	$-$	$\frac{8}{3}x_2$	$+$	$\frac{4}{3}$	$\geq$	0
D^2						y_3	$-$	$2y_4$	$+$	y_5	$\geq$	0
$D^2 C$	$-$	x_1^4	$+$	$\dots$	$+$	$4y_3$	$+$	$4y_4$	$+$	$4y_5$	$\geq$	0

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{llcrcrcrcrcrcl}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{llcrcrcrcrcrcl}
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2 C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrrr}
AB & - & y_6 & + & \ldots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \ldots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \ldots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & & \geq & 0 \\
\text{irredundant:} \\
AB & - & y_6 & + & \ldots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & & \geq & 0 \\
AC & & y_{10} & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & & \geq & 0 \\
ABC & - & y_{13} & + & \ldots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & & \geq & 0 \\
D^2C & - & x_1^4 & + & \ldots & + & 4y_3 & + & 4y_4 & + & 4y_5 & & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & & \geq & 0 \\
\text{irredundant:} \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & & \geq & 0 \\
AC & & y_{10} & + & \dots & - & x_1 & + & 8x_2 & - & 4 & & \geq & 0 \\
ABC & - & y_{13} & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & & \geq & 0
\end{array}
$$

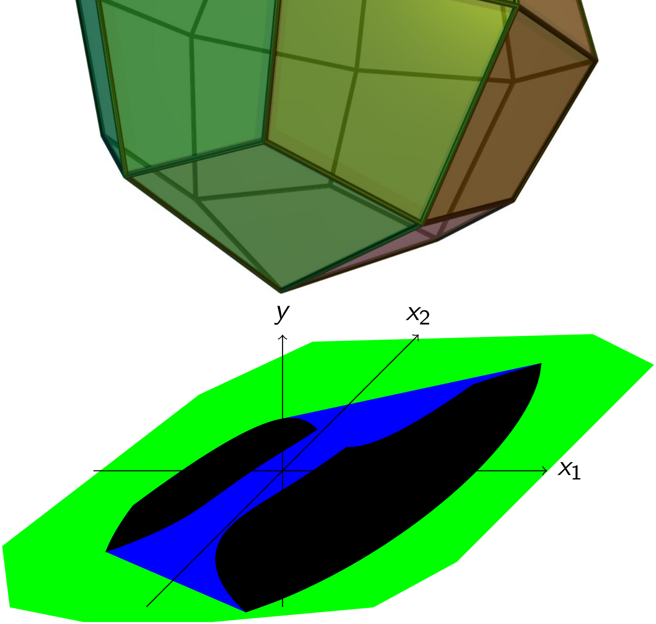
System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & & & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & & & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrrr}
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & y_{13} & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & y_{18} & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$



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The Positivstellensatz from real algebraic geometry

In his very first and fulminant work Anneaux préordonnés, Krivine proved in 1964 the so-called Positivstellensatz

1

The Positivstellensatz from real algebraic geometry

In his very first and fulminant work Anneaux préordonnés, Krivine proved in 1964 the so-called Positivstellensatz which is essentially equivalent to the following theorem:

1

The Positivstellensatz from real algebraic geometry

In his very first and fulminant work Anneaux préordonnés, Krivine proved in 1964 the so-called Positivstellensatz which is essentially equivalent to the following theorem:

To each given unsolvable (finite) system of (non-strict) polynomial inequalities (in several variables), in the just described (less naive) linearization procedure, you can always add finitely many blue inequalities such that the resulting system of linear inequalities is unsolvable.

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Schmüdgen's and Putinar's Positivstellensätze

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All proofs use the Positivstellensatz (from Krivine). Schmüdgen's original proof uses in addition functional analysis. The first purely algebraic proof was found by Wörmann in 1998.

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In 1993, Putinar showed that products like $A \cdot B$ are not needed in Schmüdgen's theorem when the compactness assumption is replaced by a stronger technical assumption, namely the archimedean condition, which is for practical purposes not far from compactness.

System of polynomial inequalities

Systematic linearization

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

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redundant families (parametrized by $a, b, c, \ldots$):

System of polynomial inequalities

Systematic linearization

$$\begin{array}{lrrrrrrl} A & & x_1^3 & - & x_1 & - & 2x_2 & + 1 \geq 0 \\ B & - x_2^4 & + 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - \frac{1}{3} \geq 0 \\ C & & - x_1^2 & - & x_2^2 & + & x_1 & + 4 \geq 0 \end{array}$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2 + dx_1^2 + ex_1x_2 + fx_2^2)^2 \geq 0$$

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$$(a+bx_1+cx_2+dx_1^2+ex_1x_2+fx_2^2)\begin{pmatrix}1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\\e\\f\end{pmatrix} \geq 0$$

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$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\ x_1 & x_1^2 & x_1x_2 & x_1^3 & x_1^2x_2 & x_1x_2^2 \\ x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ x_1^2 & x_1^3 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

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x_1 & x_1^2 & x_1x_2 & x_1^3 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & x_1^3 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
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x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
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1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
x_1 & x_1^2 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}
\geq 0
$$

System of polynomial inequalities

Systematic linearization

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1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
x_1 & x_1^2 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Systematic linearization

$$\begin{array}{lrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
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System of polynomial inequalities

Systematic linearization

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A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
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$$

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\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & x_1x_2 & x_2^2 \\
x_1 & y_3 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
y_3 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Systematic linearization

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\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & x_2^2 \\
x_1 & y_3 & y_4 & y_1 & x_1^2 x_2 & x_1 x_2^2 \\
x_2 & y_4 & x_2^2 & x_1^2 x_2 & x_1 x_2^2 & x_2^3 \\
y_3 & y_1 & x_1^2 x_2 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\
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x_2^2 & x_1 x_2^2 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

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System of polynomial inequalities

Systematic linearization

$$\begin{array}{lcccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & x_1^2 x_2 & x_1 x_2^2 \\ x_2 & y_4 & y_5 & x_1^2 x_2 & x_1 x_2^2 & x_2^3 \\ y_3 & y_1 & x_1^2 x_2 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\ y_4 & x_1^2 x_2 & x_1 x_2^2 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\ y_5 & x_1 x_2^2 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Systematic linearization

$$
\begin{array}{lccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & x_1^2 x_2 & x_1 x_2^2 \\
x_2 & y_4 & y_5 & x_1^2 x_2 & x_1 x_2^2 & x_2^3 \\
y_3 & y_1 & x_1^2 x_2 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & x_1^2 x_2 & x_1 x_2^2 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & x_1 x_2^2 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Systematic linearization

$$
\begin{array}{lcccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & x_1 x_2^2 \\
x_2 & y_4 & y_5 & y_6 & x_1 x_2^2 & x_2^3 \\
y_3 & y_1 & y_6 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & y_6 & x_1 x_2^2 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & x_1 x_2^2 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Systematic linearization

$$
\begin{array}{lcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & x_1 x_2^2 \\
x_2 & y_4 & y_5 & y_6 & x_1 x_2^2 & x_2^3 \\
y_3 & y_1 & y_6 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & y_6 & x_1 x_2^2 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & x_1 x_2^2 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Systematic linearization

$$
\begin{array}{lcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & x_2^3 \\
y_3 & y_1 & y_6 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & y_7 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Systematic linearization

$$
\begin{array}{lcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & x_2^3 \\
y_3 & y_1 & y_6 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & y_7 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Systematic linearization

$$
\begin{array}{lccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & x_2^3 \\
y_3 & y_1 & y_6 & y_8 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & y_7 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Systematic linearization

$$\begin{array}{lcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & x_2^3 \\ y_3 & y_1 & y_6 & y_8 & x_1^3 x_2 & x_1^2 x_2^2 \\ y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\ y_5 & y_7 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Systematic linearization

$$\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & y_7 & y_9 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Systematic linearization

$$
\begin{array}{lrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & y_7 & y_9 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Systematic linearization

$$\begin{array}{lcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\ y_3 & y_1 & y_6 & y_8 & y_{10} & x_1^2x_2^2 \\ y_4 & y_6 & y_7 & y_{10} & x_1^2x_2^2 & x_1x_2^3 \\ y_5 & y_7 & y_9 & x_1^2x_2^2 & x_1x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Systematic linearization

$$
\begin{array}{lrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & y_{10} & x_1^2 x_2^2 \\
y_4 & y_6 & y_7 & y_{10} & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & y_7 & y_9 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}
\geq 0
$$

System of polynomial inequalities

Systematic linearization

$$\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\ y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\ y_4 & y_6 & y_7 & y_{10} & y_{11} & x_1 x_2^3 \\ y_5 & y_7 & y_9 & y_{11} & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Systematic linearization

$$\begin{array}{lrrrrrrl}
A & & & y_1 & - x_1 & - 2x_2 & + 1 & \geq 0 \\
B & - y_2 & + 2y_3 & - 2y_4 & + y_5 & - \frac{1}{3} & & \geq 0 \\
C & & - y_3 & - y_5 & + x_1 & + 4 & & \geq 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\ y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\ y_4 & y_6 & y_7 & y_{10} & y_{11} & x_1 x_2^3 \\ y_5 & y_7 & y_9 & y_{11} & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Systematic linearization

$$
\begin{array}{lcccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\
y_4 & y_6 & y_7 & y_{10} & y_{11} & y_{12} \\
y_5 & y_7 & y_9 & y_{11} & y_{12} & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Systematic linearization

$$\begin{array}{lccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\
y_4 & y_6 & y_7 & y_{10} & y_{11} & y_{12} \\
y_5 & y_7 & y_9 & y_{11} & y_{12} & y_2
\end{pmatrix} \succeq 0$$

System of polynomial inequalities

Systematic linearization

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Systematic linearization

$$\begin{array}{lcccccccccccccc}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0$$

System of polynomial inequalities

Systematic linearization

$$
\begin{array}{lcccccccccccc}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0 \qquad \iff$$

$$(-x_1^2 - x_2^2 + x_1 + 4)(a + bx_1 + cx_2)\begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Systematic linearization

$$\begin{array}{lcccccccccccccc}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0 \qquad \Longleftrightarrow$$

$$(-x_1^2 - x_2^2 + x_1 + 4) \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Systematic linearization

$$
\begin{array}{lcccccccccccc}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0 \qquad \iff$$

$$\begin{pmatrix} a & b & c \end{pmatrix} (-x_1^2 - x_2^2 + x_1 + 4) \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Systematic linearization

$$
\begin{array}{lrrrrrrrrrrrr}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0 \qquad \Longleftrightarrow$$

$$\begin{pmatrix} a & b & c \end{pmatrix} (-x_1^2 - x_2^2 + x_1 + 4) \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & x_1^2 & x_1x_2 \\ x_2 & x_1x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Systematic linearization

$$\begin{array}{llcccccccccccc}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -x_1^2 - x_2^2 + x_1 + 4 & \ldots & \ldots \\ -x_1^3 - x_1x_2^2 + x_1^2 + 4x_1 & \ldots & \ldots \\ -x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \ldots & \ldots \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Systematic linearization

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$$

redundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-x_1^2 - x_2^2 + x_1 + 4 & \dots & \dots \\
-x_1^3 - x_1x_2^2 + x_1^2 + 4x_1 & \dots & \dots \\
-x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Systematic linearization

$$\begin{array}{lrrrrrrrrrrrrr}
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-y_1 - x_1x_2^2 + x_1^2 + 4x_1 & \ldots & \ldots \\
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\begin{pmatrix}
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\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

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$$\begin{array}{lrrrrrrrrrrrrr}
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Systematic linearization

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System of polynomial inequalities

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-x_1^2 x_2 - x_2^3 + y_4 + 4x_2 & \ldots & \ldots
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System of polynomial inequalities

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C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
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System of polynomial inequalities

Systematic linearization

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\begin{array}{lrrrrrrrrrrrrr}
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B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
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-y_7 - x_2^3 + y_4 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Systematic linearization

$$\begin{array}{llllllllllllllll} A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\ B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\ C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0 \end{array}$$

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Systematic linearization

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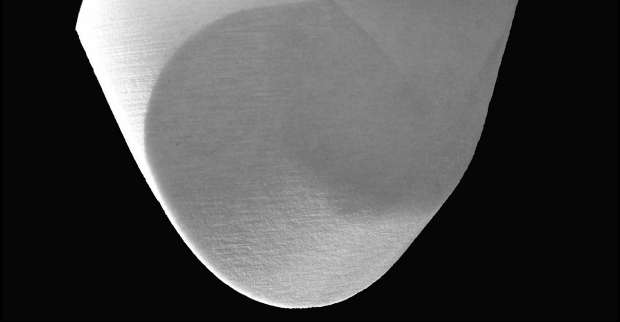

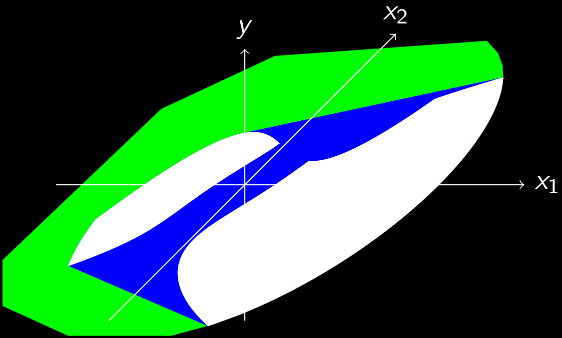

conv S

Putinar's Positivstellensatz

Denote by $\underline{X} := (\underline{X}_1, \ldots, \underline{X}_n)$ a tuple of variables, let $g_1, \ldots, g_m \in \mathbb{R}[\underline{X}]$ be polynomials and set $g_0 := 1 \in \mathbb{R}[\underline{X}]$.

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Then $S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$ is called a basic closed semialgebraic set and boolean combinations of such are called semialgebraic.

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We will always assume that $g_1, \ldots, g_m$ satisfy the archimedean condition. For the purpose of this talk, it can be thought of S being compact although it is a slightly stronger technical condition.

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Theorem (Putinar). If $f \in \mathbb{R}[\underline{X}]$ satisfies $f > 0$ on S, then there exist $p_{ij} \in \mathbb{R}[\underline{X}]$ such that

$$f = \sum_{i=0}^{m} \sum_{j} p_{ij}^2 g_i.$$

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Warning. "$>$" cannot be replaced by "$\geq$".

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Warning. Relies on degree cancellation: $\deg p_{ij} \gg \deg f$ frequently occurs.

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Warning. Even for $\ell \in \mathbb{R}[\underline{X}]$ linear, that is $\deg \ell \leq 1$: If $\ell \geq 0$ on S but ℓ has a zero on S, then for $f := \ell + \varepsilon$ the degrees of the p_{ij} might explode when $\varepsilon > 0$ tends to zero.

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$$f = \sum_{i=0}^{m} \sum_{j} p_{ij}^2 g_i.$$

Hope. If by chance, for the given $g_1, \dots, g_m$, we have a degree bound N such that for all linear f with $f \geq 0$ on S we have such a representation with $\deg p_{ij} \leq N$, then our "linearization" works perfectly well.

The ingenious idea of Helton & Nie

Let $\ell \in \mathbb{R}[\underline{X}]$ be linear, $\ell \geq 0$ on S and $u \in S$ with $\ell(u) = 0$.

The ingenious idea of Helton & Nie

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Let $\ell \in \mathbb{R}[\underline{X}]$ be linear, $\ell \geq 0$ on S and $u \in S$ with $\ell(u) = 0$. We would like to write ℓ as a weighted sum of squares of controlled degree with weights g_i. Then u is a minimizer of ℓ on $S = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$.

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Concavity and quasi-concavity

The following local second-order notions are convenient for us:
We call $g \in \mathbb{R}[\underline{X}]$ on S

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- strictly concave if $g''(x) \prec 0$ for all $x \in S$,
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- strictly quasi-concave if $v^T g''(x) v < 0$ for all $x \in S$ and $v \in \mathbb{R}^n \setminus \{0\}$ satisfying $g'(x)v = 0$.

Hol & Scherer's Positivstellensatz

Remember our task to find a "sum of squares representation" of

$$F_{i,u} = \int_{t=0}^{1} \int_{s=0}^{t} (-g_i'')(u + s(\underline{X} - u)) \in S\mathbb{R}[\underline{X}]^{n \times n}$$

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Theorem (Hol & Scherer). If $F \in S\mathbb{R}[\underline{X}]^{r\times r}$ satisfies $F \succ 0$ on S, then there exist sos matrix polynomials $P_i \in S\mathbb{R}[\underline{X}]^{r\times r}$ such that

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Warning. Should not apply it with $F = -g_i''$ since we have to deal with $(-g_i'')(u + s(\underline{X} - u)) = \sum_{i=0}^{m} g_i(u + s(\underline{X} - u)) P_i(u + s(\underline{X} - u))$ afterwards: $P_i(u + s(\underline{X} - u))$ is sos but don't know what to do with $g_i(u + s(\underline{X} - u))$.

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Idea of Helton & Nie: Apply it with $F = F_{i,u}$. Since this depends on $u \in S$ we have to take care of the degree of the representation.
$\rightsquigarrow$ Prove quantitative version of Hol & Scherer's theorem.

Helton & Nie's Positivstellensatz

The following is the needed quantitative version of Hol & Scherer:

Theorem (Helton & Nie). To any given $r, N \in \mathbb{N}$, there exists $D \in \mathbb{N}$ (depending on n, m, $g_1, \dots, g_m$, r and N) such that for all $F \in S\mathbb{R}[\underline{X}]^{r \times r}$ satisfying $\|F\| \leq N$ and $F \succeq \frac{1}{N}$ on S, there exist sos matrix polynomials $P_i \in S\mathbb{R}[\underline{X}]^{r \times r}$ of degree at most D such that

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The proof of this was initially hard but Kriel found in his master's thesis an amazingly short way of reducing this to Hol & Scherer.

Let again $i \in \{1, \dots, m\}$ and $u \in S$ with $g_i(u) = 0$ and consider

$$F_{i,u} = \int_{t=0}^{1} \int_{s=0}^{t} (-g_i'')(u + s(\underline{X} - u)) \in S\mathbb{R}[\underline{X}]^{n \times n}.$$

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If g_i is strictly quasi-concave on S, then Helton & Nie instead look at $\ell - \sum_{i=1}^{m} \lambda_i \tilde{g}_i$ where $\tilde{g}_i = g_i h(g_i)$ for a one variable polynomial $h \in \mathbb{R}[T]$ making $h(g_i) > 0$ on S, $\tilde{g}_i$ strictly concave on S and $\tilde{S} \cap U = S$ for some open set U.

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Suppose now that S has non-empty interior and each g_i is quasi-concave on $S_i := \{x \in \mathbb{R}^n \mid g_i(x) = 0\}$.

Then it is easy to see that $S_i \subseteq \partial S$ and therefore $\partial S = \bigcup_{i=1}^m S_i$.

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We had the following new idea: Establish an additional property of $h \in \mathbb{R}[T]$ so that the Hessian of $\tilde{g}_i = g_i h(g_i)$ decays rapidly in norm when moving from the boundary to the inside of S. While double-integrating on the line segment in

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you accumulate very close to the boundary so much positive-definiteness that some of it will stay until you arrive at the inner end of the segment.

Main Theorem

Theorem (Kriel & S.) Suppose S is convex with non-empty interior and each g_i is sos-convex or quasi-concave on S_i. Then the moment relaxations of sufficiently high degree are exact.

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Outlook

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Theorem (Scheiderer). True for $n = 2$.

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