Positive polynomials and optimization

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HPOPT 2004

Centrum voor Wiskunde en Informatica

Amsterdam, June 23, 2004

Part I: Representation of positive polynomials

Part II: Optimization of polynomials

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on compact basic closed semialgebraic sets

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- $g_0 := 1 \in \mathbb{R}[X]$ for convenience

n

f

$g_1, \dots, g_m$

S

g_0

Example. When

- $m = n$
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then S is the nonnegative orthant $[0, \infty)^n$.

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- $m = n + 1$
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- $g_{n+1} := 1 - (X_1 + \cdots + X_n)$

then S is the unit simplex in $\mathbb{R}^n$.

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- $m = n + 2$
- $g_1 := X_1, \ldots, g_n := X_n$
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- $g_{n+2} := X_1 + \cdots + X_n - 1$

then S is the standard simplex

$$\Delta_n := \{x \in \mathbb{R}^n \mid x_1 \geq 0, \ldots, x_n \geq 0, x_1 + \cdots + x_n = 1\}.$$

Theorem of Pólya.

George Pólya: Über positive Darstellung von Polynomen
Vierteljahresschrift der Naturforschenden Ges. in Zürich **73**, 141–145 (1928)

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$$(X_1 + \cdots + X_n)^k f = \sum_{|\alpha|=k+d} \frac{k!(k+d)^d}{\alpha_1! \ldots \alpha_n!} f \quad \underbrace{\left(\frac{\alpha}{k+d}\right)}_{\in \Delta_n} X^\alpha$$

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$$f_\varepsilon := \sum_{|\alpha|=d} a_\alpha (X_1)_\varepsilon^{\alpha_1} \cdots (X_n)_\varepsilon^{\alpha_n} \qquad \text{where} \qquad (X_i)_\varepsilon^{\alpha_i} := \prod_{j=0}^{\alpha_i - 1} (X_i - j\varepsilon).$$

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But $f_\varepsilon \to f$ uniformly on Δ_n for $\varepsilon \to 0$.

Sums of squares and quadratic modules

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&= \left\{\sigma g_i \mid \sigma \in \sum \mathbb{R}[X]^2\right\} \qquad \text{and so on} \ldots
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is a set of polynomials which are

for obvious reasons ≥ 0 on S.

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is the quadratic module generated by $g_1, \ldots, g_m$.

Quadratic module: $0, 1 \in M$, $M + M \subseteq M$ and $\mathbb{R}[X]^2 M \subseteq M$.

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Mihai Putinar: Positive polynomials on compact semi-algebraic sets
Indiana Univ. Math. J. **42**, No. 3, 969–984 (1993)

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- Several non-trivial sufficient criteria for the hypothesis being satisfied due to Schmüdgen and Jacobi & Prestel.

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- Substitute back old coordinates.

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Lifting Lemma. Suppose S and $C \subseteq \mathbb{R}^n$ are compact, $g_i \leq 1$ on C.
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Suppose $f > 0$ on S. Then $\exists s \in \mathbb{N} : \forall$ sufficiently large $k \in \mathbb{N}$:

$$f - s \sum_{i=1}^{m} \underbrace{(1 - g_i)^{2k} g_i} > 0 \qquad \text{on } C.$$

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Lifting Lemma. Suppose S and $C \subseteq \mathbb{R}^n$ are compact, $g_i \leq 1$ on C.
Suppose $f > 0$ on S. Then $\exists s \in \mathbb{N} : \forall$ sufficiently large $k \in \mathbb{N}$:

$$f - s \sum_{i=1}^{m} \underbrace{(1 - g_i)^{2k} g_i}_{\in M} > 0 \qquad \text{on } C.$$

Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^{n} X_i^2 \in M$.
Then $f > 0$ on $S \implies f \in M$.
Proof outline.

- WLOG $f > 0$ on an arbitrary fixed compact set $C \subseteq \mathbb{R}^n$.
  Find $h \in M$ such that $f - h > 0$ on C. If we show $f - h \in M$, then $f \in M$.

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$$f - s \sum_{i=1}^{m} \underbrace{(1 - g_i)^{2k} g_i}_{\substack{\to 0+ \\ \text{when } g_i \geq 0}} > 0 \qquad \text{on } C.$$

Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^{n} X_i^2 \in M$.
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Then $f > 0$ on $S \implies f \in M$.

Proof. WLOG $f > 0$ on the compact set $C := ?$.

Lifting Lemma. Suppose S and $C \subseteq \mathbb{R}^n$ are compact, $g_i \leq 1$ on C.
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Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^{n} X_i^2 \in M$.

Then $f > 0$ on $S \implies f \in M$.

Proof. WLOG $f > 0$ on the compact set $C :=$?.

Show even that $f \in T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 \left(N - \sum_{i=1}^{n} X_i^2\right)$.

$\rightsquigarrow$ Note $T \subseteq M$. Advantage: $TT \subseteq T$.

Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^{n} X_i^2 \in M$.
Then $f > 0$ on $S \implies f \in M$.

Proof. WLOG $f > 0$ on the compact set $C := \bigcirc$.
Show even that $f \in T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 \left(N - \sum_{i=1}^{n} X_i^2\right)$.
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Proof. WLOG $f > 0$ on the compact set $C := \bigcirc$.

Show even that $f \in T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 \left(N - \sum_{i=1}^{n} X_i^2\right)$.

$\leadsto$ Note $T \subseteq M$. Advantage: $TT \subseteq T$.

Choose n new coordinates lying in T

$$Y_i \mapsto \left(N + \frac{1}{4}\right) + X_i = \sum_{j \neq i} X_j^2 + \left(X_i + \frac{1}{2}\right)^2 + \left(N - \sum_{j=1}^{n} X_j^2\right) \in T$$

Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^n X_i^2 \in M$.
Then $f > 0$ on $S \implies f \in M$.

Proof. WLOG $f > 0$ on the compact set $C := \bigcirc$.

Show even that $f \in T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 \left(N - \sum_{i=1}^n X_i^2\right)$.

$\rightsquigarrow$ Note $T \subseteq M$. Advantage: $TT \subseteq T$.

Choose $2n$ new coordinates lying in T

$$Y_{\substack{i \\ i+n}} \mapsto \left(N + \frac{1}{4}\right) \pm X_i = \sum_{j \neq i} X_j^2 + \left(X_i \pm \frac{1}{2}\right)^2 + \left(N - \sum_{j=1}^n X_j^2\right) \in T$$

and summing up to a constant.

Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^{n} X_i^2 \in M$.

Then $f > 0$ on $S \implies f \in M$.

Proof. WLOG $f > 0$ on the compact set $C := \bigcirc$.

Show even that $f \in T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 \left(N - \sum_{i=1}^{n} X_i^2\right)$.

$\rightsquigarrow$ Note $T \subseteq M$. Advantage: $TT \subseteq T$.

Choose $2n$ new coordinates lying in T

$$\underset{i+n}{Y_{\ i}} \mapsto \left(N + \frac{1}{4}\right) \underset{-}{+} X_i = \sum_{j \neq i} X_j^2 + \left(X_i \underset{-}{+} \frac{1}{2}\right)^2 + \left(N - \sum_{j=1}^{n} X_j^2\right) \in T$$

and summing up to $2n(N + \frac{1}{4})$.

Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^{n} X_i^2 \in M$.
Then $f > 0$ on $S \implies f \in M$.

Proof. WLOG $f > 0$ on the compact set $C := \bigcirc$.

Show even that $f \in T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 \left(N - \sum_{i=1}^{n} X_i^2\right)$.

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$$Y_{\substack{i \\ i+n}} \mapsto \left(N + \frac{1}{4}\right) \pm X_i = \sum_{j \neq i} X_j^2 + \left(X_i \pm \frac{1}{2}\right)^2 + \left(N - \sum_{j=1}^{n} X_j^2\right) \in T$$

and summing up to $2n(N + \frac{1}{4})$.

A way of writing f in the new coordinates:

$$f\left(\underbrace{\frac{Y_1 - Y_{n+1}}{2}}, \ldots, \underbrace{\frac{Y_n - Y_{2n}}{2}}\right)$$

Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^{n} X_i^2 \in M$.
Then $f > 0$ on $S \implies f \in M$.

Proof. WLOG $f > 0$ on the compact set $C := \bigcirc$.
Show even that $f \in T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 \left(N - \sum_{i=1}^{n} X_i^2\right)$.
$\rightsquigarrow$ Note $T \subseteq M$. Advantage: $TT \subseteq T$.
Choose $2n$ new coordinates lying in T

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and summing up to $2n(N + \frac{1}{4})$.

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in the new coordinates:

$$f\left(\underbrace{\frac{Y_1 - Y_{n+1}}{2}}_{\mapsto X_1}, \dots, \underbrace{\frac{Y_n - Y_{2n}}{2}}\right)$$

Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^n X_i^2 \in M$.
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Show even that $f \in T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 \left(N - \sum_{i=1}^n X_i^2\right)$.

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Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^{n} X_i^2 \in M$.

Then $f > 0$ on $S \implies f \in M$.

Proof. WLOG $f > 0$ on the compact set $C := \bigcirc$.

Show even that $f \in T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 \left(N - \sum_{i=1}^{n} X_i^2\right)$.

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Choose $2n$ new coordinates lying in T

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and summing up to $2n(N + \frac{1}{4})$. Write $f = \sum_{i=0}^{d} f_i$, f_i homogeneous of degree i.

A way of writing f in the new coordinates:

$$f\left(\underbrace{\frac{Y_1 - Y_{n+1}}{2}}_{\mapsto X_1}, \ldots, \underbrace{\frac{Y_n - Y_{2n}}{2}}_{\mapsto X_n}\right)$$

Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^{n} X_i^2 \in M$.

Then $f > 0$ on $S \implies f \in M$.

Proof. WLOG $f > 0$ on the compact set $C := \bigcirc$.

Show even that $f \in T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 \left(N - \sum_{i=1}^{n} X_i^2\right)$.

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Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^n X_i^2 \in M$. Then $f > 0$ on $S \implies f \in M$.

Proof. WLOG $f > 0$ on the compact set $C := \bigcirc$.

Show even that $f \in T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 \left(N - \sum_{i=1}^n X_i^2\right)$.

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and summing up to $2n(N + \frac{1}{4})$. Write $f = \sum_{i=0}^d f_i$, f_i homogeneous of degree i.

A way of writing f in the new coordinates:

$$\sum_{i=0}^d f_i \Bigg(\underbrace{\frac{Y_1 - Y_{n+1}}{2}}_{\mapsto X_1}, \dots, \underbrace{\frac{Y_n - Y_{2n}}{2}}_{\mapsto X_n} \Bigg) \Bigg(\underbrace{\frac{Y_1 + \dots + Y_{2n}}{2n(N + \frac{1}{4})}} \Bigg)^{d-i}$$

Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^{n} X_i^2 \in M$.
Then $f > 0$ on $S \implies f \in M$.

Proof. WLOG $f > 0$ on the compact set $C := \bigcirc$.

Show even that $f \in T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 \left(N - \sum_{i=1}^{n} X_i^2\right)$.

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Choose $2n$ new coordinates lying in T

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Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^{n} X_i^2 \in M$.
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Choose $2n$ new coordinates lying in T

$$Y_{\substack{i \\ i+n}} \mapsto \left(N + \frac{1}{4}\right) \pm X_i = \sum_{j \neq i} X_j^2 + \left(X_i \pm \frac{1}{2}\right)^2 + \left(N - \sum_{j=1}^{n} X_j^2\right) \in T$$

and summing up to $2n(N + \frac{1}{4})$. Set $\Delta := \left\{ y \mid \substack{y_1 \geq 0, \ldots, y_{2n} \geq 0 \\ \sum y_i = 2n(N + \frac{1}{4})} \right\}$.

Apply Pólya on
$$\sum_{i=0}^{d} f_i \Bigg(\underbrace{\frac{Y_1 - Y_{n+1}}{2}}_{\mapsto X_1}, \ldots, \underbrace{\frac{Y_n - Y_{2n}}{2}}_{\mapsto X_n} \Bigg) \Bigg(\underbrace{\frac{Y_1 + \cdots + Y_{2n}}{2n(N + \frac{1}{4})}}_{\mapsto 1} \Bigg)^{d-i}$$

Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^{n} X_i^2 \in M$.
Then $f > 0$ on $S \implies f \in M$.

Proof. WLOG $f > 0$ on the compact set $C := \bigcirc$.
Show even that $f \in T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 \left(N - \sum_{i=1}^{n} X_i^2\right)$.
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Choose $2n$ new coordinates lying in T

$$Y_{\substack{i \\ i+n}} \mapsto \left(N + \frac{1}{4}\right) \underset{-}{+} X_i = \sum_{j \neq i} X_j^2 + \left(X_i \underset{-}{+} \frac{1}{2}\right)^2 + \left(N - \sum_{j=1}^{n} X_j^2\right) \in T$$

and summing up to $2n(N + \frac{1}{4})$. Set $\Delta := \left\{y \mid \substack{y_1 \geq 0, \ldots, y_{2n} \geq 0 \\ \sum y_i = 2n(N + \frac{1}{4})}\right\}$.

Apply Pólya on
$$\sum_{i=0}^{d} f_i \Bigg(\underbrace{\frac{Y_1 - Y_{n+1}}{2}}_{\mapsto X_1}, \ldots, \underbrace{\frac{Y_n - Y_{2n}}{2}}_{\mapsto X_n}\Bigg) \Bigg(\underbrace{\overbrace{\frac{Y_1 + \cdots + Y_{2n}}{2n(N + \frac{1}{4})}}^{=1 \text{ on } \Delta}}_{\mapsto 1}\Bigg)^{d-i}$$

Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^{n} X_i^2 \in M$.
Then $f > 0$ on $S \implies f \in M$.

Proof. WLOG $f > 0$ on the compact set $C := \bigcirc$.

Show even that $f \in T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 \left(N - \sum_{i=1}^{n} X_i^2\right)$.

$\rightsquigarrow$ Note $T \subseteq M$. Advantage: $TT \subseteq T$.

Choose $2n$ new coordinates lying in T

$$\underset{i+n}{Y_{\;i}} \mapsto \left(N + \frac{1}{4}\right) \pm X_i = \sum_{j \neq i} X_j^2 + \left(X_i \pm \frac{1}{2}\right)^2 + \left(N - \sum_{j=1}^{n} X_j^2\right) \in T$$

and summing up to $2n(N + \frac{1}{4})$. Set $\Delta := \left\{y \mid {y_1 \geq 0, \ldots, y_{2n} \geq 0 \atop \sum y_i = 2n(N + \frac{1}{4})}\right\}$.

Apply Pólya on

$$\overbrace{\sum_{i=0}^{d} f_i \Bigg(\underbrace{\frac{Y_1 - Y_{n+1}}{2}}_{\mapsto X_1}, \ldots, \underbrace{\frac{Y_n - Y_{2n}}{2}}_{\mapsto X_n} \Bigg) \Bigg(\underbrace{\overbrace{\frac{Y_1 + \cdots + Y_{2n}}{2n(N + \frac{1}{4})}}^{=1 \text{ on } \Delta}}_{\mapsto 1} \Bigg)^{d-i}}^{> 0 \text{ on } \Delta \text{ when } f > 0 \text{ on } \Diamond := l(\Delta)}$$

Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^n X_i^2 \in M$.
Then $f > 0$ on $S \implies f \in M$.

Proof. WLOG $f > 0$ on the compact set $C := \bigcirc$.

Show even that $f \in T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 \left(N - \sum_{i=1}^n X_i^2\right)$.

$\rightsquigarrow$ Note $T \subseteq M$. Advantage: $TT \subseteq T$.

Choose $2n$ new coordinates lying in T

$$\underset{i+n}{Y_i} \mapsto \left(N + \frac{1}{4}\right) \underset{-}{+} X_i = \sum_{j \neq i} X_j^2 + \left(X_i \underset{-}{+} \frac{1}{2}\right)^2 + \left(N - \sum_{j=1}^n X_j^2\right) \in T$$

and summing up to $2n(N + \frac{1}{4})$. Set $\Delta := \left\{y \mid \substack{y_1 \geq 0, \dots, y_{2n} \geq 0 \\ \sum y_i = 2n(N+\frac{1}{4})}\right\}$.

Apply Pólya on

$$\overbrace{\sum_{i=0}^d f_i \overbrace{\Bigg(\underbrace{\frac{Y_1 - Y_{n+1}}{2}}_{\mapsto X_1}, \dots, \underbrace{\frac{Y_n - Y_{2n}}{2}}_{\mapsto X_n}\Bigg)}^{\text{defines linear map } l : \mathbb{R}^{2n} \to \mathbb{R}^n}}^{> 0 \text{ on } \Delta \text{ when } f > 0 \text{ on } \Diamond := l(\Delta)} \Bigg(\overbrace{\underbrace{\frac{Y_1 + \dots + Y_{2n}}{2n(N + \frac{1}{4})}}_{\mapsto 1}}^{=1 \text{ on } \Delta}\Bigg)^{d-i}$$

Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^n X_i^2 \in M$.
Then $f > 0$ on $S \implies f \in M$.

Proof. WLOG $f > 0$ on the compact set $C := \Diamond$.

Show even that $f \in T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 \left(N - \sum_{i=1}^n X_i^2\right)$.

$\rightsquigarrow$ Note $T \subseteq M$. Advantage: $TT \subseteq T$.

Choose $2n$ new coordinates lying in T

$$Y_{\substack{i \\ i+n}} \mapsto \left(N + \frac{1}{4}\right) \pm X_i = \sum_{j \neq i} X_j^2 + \left(X_i \pm \frac{1}{2}\right)^2 + \left(N - \sum_{j=1}^n X_j^2\right) \in T$$

and summing up to $2n(N + \frac{1}{4})$. Set $\Delta := \left\{y \mid \substack{y_1 \geq 0, \ldots, y_{2n} \geq 0 \\ \sum y_i = 2n(N + \frac{1}{4})}\right\}$.

Apply Pólya on

$$\overbrace{\sum_{i=0}^d f_i \Big(\overbrace{\underbrace{\frac{Y_1 - Y_{n+1}}{2}}_{\mapsto X_1}, \ldots, \underbrace{\frac{Y_n - Y_{2n}}{2}}_{\mapsto X_n}}^{\text{defines linear map } l : \mathbb{R}^{2n} \to \mathbb{R}^n} \Big)}^{> 0 \text{ on } \Delta \text{ since } f > 0 \text{ on } \Diamond := l(\Delta)} \Bigg(\overbrace{\underbrace{\frac{Y_1 + \cdots + Y_{2n}}{2n(N + \frac{1}{4})}}_{\mapsto 1}}^{=1 \text{ on } \Delta} \Bigg)^{d-i}$$

Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^n X_i^2 \in M$.
Then $f > 0$ on $S \implies f \in M$.

Proof. WLOG $f > 0$ on the compact set $C := \Diamond$.

Show even that $f \in T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 \left(N - \sum_{i=1}^n X_i^2\right)$.

$\rightsquigarrow$ Note $T \subseteq M$. Advantage: $TT \subseteq T$.

Choose $2n$ new coordinates lying in T

$$\underset{i+n}{Y_{\ i}} \mapsto \left(N + \frac{1}{4}\right) \underset{-}{+} X_i = \sum_{j \neq i} X_j^2 + \left(X_i \underset{-}{+} \frac{1}{2}\right)^2 + \left(N - \sum_{j=1}^n X_j^2\right) \in T$$

and summing up to $2n(N + \frac{1}{4})$. Set $\Delta := \{y \mid {}^{y_1 \geq 0, \ldots, y_{2n} \geq 0}_{\sum y_i = 2n(N + \frac{1}{4})}\}$.

$$\left(\frac{Y_1 + \cdots + Y_{2n}}{}\right)^k \overbrace{\sum_{i=0}^d f_i \overbrace{\Bigg(\underbrace{\frac{Y_1 - Y_{n+1}}{2}}_{\mapsto X_1}, \ldots, \underbrace{\frac{Y_n - Y_{2n}}{2}}_{\mapsto X_n}\Bigg)}^{\text{defines linear map } l : \mathbb{R}^{2n} \to \mathbb{R}^n}}^{> 0 \text{ on } \Delta \text{ since } f > 0 \text{ on } \Diamond := l(\Delta)} \Bigg(\overbrace{\underbrace{\frac{Y_1 + \cdots + Y_{2n}}{2n(N + \frac{1}{4})}}_{\mapsto 1}}^{=1 \text{ on } \Delta}\Bigg)^{d-i}$$

Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^n X_i^2 \in M$.
Then $f > 0$ on $S \implies f \in M$.

Proof. WLOG $f > 0$ on the compact set $C := \lozenge$.
Show even that $f \in T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 \left(N - \sum_{i=1}^n X_i^2\right)$.
$\rightsquigarrow$ Note $T \subseteq M$. Advantage: $TT \subseteq T$.
Choose $2n$ new coordinates lying in T

$$\underset{i+n}{Y_{\,i}} \mapsto \left(N + \frac{1}{4}\right) \pm X_i = \sum_{j \neq i} X_j^2 + \left(X_i \pm \frac{1}{2}\right)^2 + \left(N - \sum_{j=1}^n X_j^2\right) \in T$$

and summing up to $2n(N + \frac{1}{4})$. Set $\Delta := \left\{y \mid \substack{y_1 \geq 0, \dots, y_{2n} \geq 0 \\ \sum y_i = 2n(N + \frac{1}{4})}\right\}$.

$$\underbrace{\left(\frac{Y_1 + \cdots + Y_{2n}}{2n(N + \frac{1}{4})}\right)}^{k} \overbrace{\sum_{i=0}^{d} f_i \overbrace{\left(\underbrace{\frac{Y_1 - Y_{n+1}}{2}}_{\mapsto X_1}, \dots, \underbrace{\frac{Y_n - Y_{2n}}{2}}_{\mapsto X_n}\right)}^{\text{defines linear map } l : \mathbb{R}^{2n} \to \mathbb{R}^n}}^{> 0 \text{ on } \Delta \text{ since } f > 0 \text{ on } \lozenge := l(\Delta)} \left(\overbrace{\underbrace{\frac{Y_1 + \cdots + Y_{2n}}{2n(N + \frac{1}{4})}}_{\mapsto 1}}^{=1 \text{ on } \Delta}\right)^{d-i}$$

Putinar's Positivstellensatz. Suppose $\exists N \in \mathbb{N} : N - \sum_{i=1}^{n} X_i^2 \in M$.
Then $f > 0$ on $S \implies f \in M$.

Proof. WLOG $f > 0$ on the compact set $C := \lozenge$.

Show even that $f \in T := \sum \mathbb{R}[X]^2 + \sum \mathbb{R}[X]^2 \left(N - \sum_{i=1}^{n} X_i^2\right)$.

$\rightsquigarrow$ Note $T \subseteq M$. Advantage: $TT \subseteq T$.

Choose $2n$ new coordinates lying in T

$$\underset{i+n}{Y_{\,i}} \mapsto \left(N + \frac{1}{4}\right) \pm X_i = \sum_{j \neq i} X_j^2 + \left(X_i \pm \frac{1}{2}\right)^2 + \left(N - \sum_{j=1}^{n} X_j^2\right) \in T$$

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Quadratic module: $0, 1 \in M$, $M + M \subseteq M$ and $\mathbb{R}[X]^2 M \subseteq M$.

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$T := \sum_{\delta \in \{0,1\}^m} \sum \mathbb{R}[X]^2 g_1^{\delta_1} \cdots g_m^{\delta_m}$ is the preorder generated by $g_1, \dots, g_m$.

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A preorder is a quadratic module closed under multiplication.

$m = 1 \implies M = T$

Krivine's Positivstellensatz. $f > 0$ on $S \iff \exists t \in T : tf \in 1 + T$

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Krivine's Positivstellensatz. $f > 0$ on $S \iff \exists t \in T : tf \in 1 + T$

- Proof is much harder.
- Is not suited for optimization problems
  but only for feasibility problems.

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Schmüdgen's Positivstellensatz. Suppose S is compact.
Then $f > 0$ on $S \implies f \in T$.

Konrad Schmüdgen: The K-moment problem for compact semi–algebraic sets
Math. Ann. **289**, No. 2, 203–206 (1991)

Schmüdgen's Positivstellensatz. Suppose S is compact.
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- Up to that our proof via Pólya works again.

Thorsten Wörmann: Strikt positive Polynome in der semialgebraischen Geometrie
Dissertation, Universität Dortmund (1998)

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- Up to that we have in this case another more complicated proof via Pólya yielding an interesting bound on the Pólya-exponent.

An algorithmic approach to Schmüdgen's Positivstellensatz
Journal of Pure and Applied Algebra **166**, 307–319 (2002)
Thorsten Wörmann: Strikt positive Polynome in der semialgebraischen Geometrie
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$$\exists N \in \mathbb{N} : N - \sum_{i=1}^{n} X_i^2 \in T \iff S \text{ compact}$$

follows either from Schmüdgen's Positivstellensatz

or by Wörmann's trick from Krivine's Positivstellensatz

$\exists N \in \mathbb{N} : N - \sum_{i=1}^{n} X_i^2 \in T \iff S$ compact

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From this follows that $\exists N \in \mathbb{N} : N - \sum_{i=1}^{n} X_i^2 \in M$ is for example equivalent to the compactness of $\{x \in \mathbb{R}^n \mid h(x) \geq 0\}$ for some $h \in M$.

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Have we cheated?

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Have we cheated?

Jacobi and Prestel: Deeper criteria.

Thomas Jacobi, Alexander Prestel:
Distinguished representations of strictly positive polynomials
J. Reine Angew. Math. **532**, 223–235 (2001)

Part II

Optimization of polynomials

on compact basic closed semialgebraic sets

Notation for the whole talk

- $X := (X_1, \dots, X_n)$ variables
- $\mathbb{R}[X] := \mathbb{R}[X_1, \dots, X_n]$ polynomial ring
- $f \in \mathbb{R}[X]$ an arbitrary polynomial
- $g_1, \dots, g_m \in \mathbb{R}[X]$ polynomials defining ...
- ... the set $S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$
- $g_0 := 1 \in \mathbb{R}[X]$ for convenience
- $M := \sum_{i=0}^{m} \sum \mathbb{R}[X]^2 g_i = \left\{\sum_{i=0}^{m} \sigma_i g_i \mid \sigma_i \in \sum \mathbb{R}[X]^2\right\}$
  the quadratic module generated by $g_1, \dots, g_m$

n

f

$g_1, \ldots, g_m$

S

g_0

M

For the rest of the talk, assume that

$$N - \sum_{i=1}^{n} X_i^2 \in M$$

for some $N \in \mathbb{N}$.

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for some $N \in \mathbb{N}$.

In particular, S is compact.

Optimization

We consider the problem of minimizing f on S.

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$$f^* := \inf\{f(x) \mid x \in S\} \in \mathbb{R} \cup \{\infty\}$$

Optimization

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$$f^* := \inf\{f(x) \mid x \in S\} \in \mathbb{R} \cup \{\infty\}$$

and, if possible, a minimizer, i.e., an element of the set

$$S^* := \{x^* \in S \mid \forall x \in S : f(x^*) \leq f(x)\}.$$

Convexification

Convexify the problem by brute force.

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- Generalize from points to probability measures:

$$f^* = \inf \left\{ \int f d\mu \mid \mu \in \mathcal{M}^1(S) \right\}$$

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Convexify the problem by brute force. Two ways to do so:

- Generalize from points to probability measures:

$$f^* = \inf \left\{ \int f d\mu \mid \mu \in \mathcal{M}^1(S) \right\}$$

- Take a dual standpoint:

$$f^* = \sup\{a \in \mathbb{R} \mid f - a \geq 0 \text{ on } S\} = \sup\{a \in \mathbb{R} \mid f - a > 0 \text{ on } S\}$$

Describing measures and positive polynomials

Putinar's solution to the moment problem. For every map $L : \mathbb{R}[X] \to \mathbb{R}$ are equivalent:

(1) L is linear, $L(1) = 1$ and $L(M) \subseteq [0, \infty)$

(2) $\exists \mu \in \mathcal{M}^1(S) : \forall p \in \mathbb{R}[X] : L(p) = \int p d\mu$

Mihai Putinar: Positive polynomials on compact semi-algebraic sets
Indiana Univ. Math. J. **42**, No. 3, 969–984 (1993)

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Stone-Weiserstrass Approximation $\Uparrow$ Riesz Representation

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$$f^* = \inf \left\{ \int f d\mu \mid \mu \in \mathcal{M}^1(S) \right\}$$

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$$\boxed{f^* = \inf\{L(f) \mid L : \mathbb{R}[X] \to \mathbb{R} \text{ is linear}, L(1) = 1, L(M) \subseteq [0, \infty)\}}$$

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Putinar's $\Downarrow$ Positivstellensatz

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$\mathbb{R}[X]$ polynomial ring

$$M := \sum_{i=0}^{m} \sum \mathbb{R}[X]^2 \; g_i$$ quadratic module

$$= \left\{ \sum_{i=0}^{m} \sigma_i g_i \mid \sigma_i \in \sum \mathbb{R}[X]^2 \right\}$$

Introduce finite-dimensional analogues $M_k \subseteq \mathbb{R}[X]_k$ of $M \subseteq \mathbb{R}[X]$.

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Here $d_i := \max\{e \in \mathbb{N} \mid 2e + \deg g_i \leq k\}$.

Warning: Never confuse M_k with $M \cap \mathbb{R}[X]_k \supseteq M_k$.

We saw that

$$f^* = \inf\{L(f) \mid L : \mathbb{R}[X] \to \mathbb{R} \text{ is linear}, L(1) = 1, L(M) \subseteq [0, \infty)\} \qquad \text{and}$$

$$f^* = \sup\{a \in \mathbb{R} \mid f - a \in M\}.$$

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In analogy to this, we set

$$P_k^* = \inf\{L(f) \mid L : \mathbb{R}[X]_k \to \mathbb{R} \text{ is linear}, L(1) = 1, L(M_k) \subseteq [0, \infty)\} \quad \text{and}$$

$$D_k^* = \sup\{a \in \mathbb{R} \mid f - a \in M_k\}$$

for every $k \in \mathcal{N}$.

We saw that

$f^* = \inf\{L(f) \mid L : \mathbb{R}[X] \to \mathbb{R} \text{ is linear}, L(1) = 1, L(M) \subseteq [0, \infty)\}$ and

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In analogy to this, we set

$P_k^* = \inf\{L(f) \mid L : \mathbb{R}[X]_k \to \mathbb{R} \text{ is linear}, L(1) = 1, L(M_k) \subseteq [0, \infty)\}$ and

$D_k^* = \sup\{a \in \mathbb{R} \mid f - a \in M_k\}$

for every $k \in \mathcal{N}$.

$P_k^* \in \mathbb{R} \cup \{\pm\infty\}$ and $D_k^* \in \mathbb{R} \cup \{\pm\infty\}$ are the optimal values of the following pair of optimization problems . . .

$$
\begin{array}{llll}
(P_k) & \text{minimize} & L(f) & \text{subject to} & L : \mathbb{R}[X]_k \to \mathbb{R} \text{ is linear,} \\
& & & & L(1) = 1 \text{ and} \\
& & & & L(M_k) \subseteq [0, \infty) \\
(D_k) & \text{maximize} & a & \text{subject to} & a \in \mathbb{R} \text{ and} \\
& & & & f - a \in M_k
\end{array}
$$

$$
\begin{array}{llll}
(P_k) & \text{minimize} & L(f) & \text{subject to} \quad L : \mathbb{R}[X]_k \to \mathbb{R} \text{ is linear,} \\
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& & & \qquad\qquad\quad L(M_k) \subseteq [0, \infty) \\
(D_k) & \text{maximize} & a & \text{subject to} \quad a \in \mathbb{R} \text{ and} \\
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\end{array}
$$

Theorem (Lasserre). $(D_k^*)_{k \in \mathcal{N}}$ and $(P_k^*)_{k \in \mathcal{N}}$ are increasing sequences that converge to f^* and satisfy $D_k^* \leq P_k^* \leq f^*$.

Proof.

$$
\begin{array}{lllll}
(P_k) & \text{minimize} & L(f) & \text{subject to} & L : \mathbb{R}[X]_k \to \mathbb{R} \text{ is linear,} \\
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(D_k) & \text{maximize} & a & \text{subject to} & a \in \mathbb{R} \text{ and} \\
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$$

Theorem (Lasserre). $(D_k^*)_{k \in \mathcal{N}}$ and $(P_k^*)_{k \in \mathcal{N}}$ are increasing sequences that converge to f^* and satisfy $D_k^* \leq P_k^* \leq f^*$.

Proof. $P_k^* \leq f^*$ because $p \mapsto p(x)$ feasible for (P_k) for $x \in S$.

$$
\begin{array}{llll}
(P_k) & \text{minimize} & L(f) & \text{subject to} & L : \mathbb{R}[X]_k \to \mathbb{R} \text{ is linear,} \\
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Jean Lasserre: Global optimization with polynomials and the problem of moments
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Theorem. Suppose $m = 1$ and $g := g_1$. Then there exists $C \in \mathbb{N}$ depending on f and g and $c \in \mathbb{N}$ depending on g such that

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f^* - D_k^* \leq \frac{C}{\sqrt[c]{k}} \qquad \text{for big } k.
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On the complexity of Schmüdgen's Positivstellensatz
Journal of Complexity **20**, No. 4, 529—543 (2004)

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Dependance on f can be made explicit. Proof hints to make dependance on g explicit for concrete g.

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In practice: Convergence usually very fast,
often $D_k^* = P_k^* = f^*$ for small k.

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What can we know from Putinar's solution to the moment problem?

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Theorem. Suppose that L_k solves (P_k) nearly to optimality $(k \in \mathcal{N})$.

$$\forall e \in \mathbb{N} : \forall \varepsilon > 0 : \exists k_0 \in \mathcal{N} \cap [e, \infty) : \forall k \geq k_0 : \exists \mu \in \mathcal{M}^1(S^*) :$$
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Optimization of polynomials on compact semialgebraic sets preprint

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Theorem (Lasserre). If S has nonempty interior, then $D_k^* = P_k^*$.

- "Strong duality"

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Optimization of polynomials on compact semialgebraic sets

preprint

Murray Marshall: Optimization of polynomial functions
Canad. Math. Bull. **46**, 575–587 (2003)
Jean Lasserre: Global optimization with polynomials and the problem of moments
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Then

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- b does not depend polynomially on $(\mathcal{D}, k)$.

Further properties of the method

- Feasible solutions of the semidefinite program corresponding to (D_k) give rise to a lower bound a of f^* together with a certificate (advantage) in form of a representation of $f - a$ proving $f - a \in M_k$.

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Optimization of polynomials on basic closed semialgebraic sets
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http://www.mathe.uni-konstanz.de/schweigh/

Announcement

Positive Polynomials

March 14–18, 2005, CIRM, Luminy (near Marseille)

organized by Marie-Françoise Roy and Markus Schweighofer

sponsored by European network RAAG and CIRM

Real Algebraic Geometry (Sums of squares)

+

Funtional Analysis (Moment problems)

+

Optimization (Interior Point Methods)

`http://www.mathe.uni-konstanz.de/schweigh/luminy2005/`

Contact: `Markus.Schweighofer@uni-konstanz.de`