Complexity aspects of SDP relaxations of polynomial optimization problems

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of polynomials which are "certifiably nonnegative on S" with "degree k sos certificates".

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Question: How good are the approximations?

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Note that now $P_k^* \geq P_{k+1}^* \geq \ldots \geq P^*$ since we maximize.

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(P_{2n}) is exact for all graphs, i.e., $P_{2n}^* = P^*$. This is not hard to show but it yields of course no polynomial time algorithm for MAXCUT since the size of (P_{2n}) grows too fast with n.

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Unique Label Cover Problem: Given a set of colors and a bipartite graph whose edges are labeled by permutations of the colors, assign colors to the nodes. Say an edge is "satisfied" if the coloring "respects" the corresponding permutation.

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But how to prove it?

It seems very difficult to generalize the random hyperplane rounding.

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Convergence of SDP relaxations

Theorem (joint with Jiawang Nie): If $(*)$ holds and $S \neq \emptyset$, then there is

- a constant $c > 0$ depending only on $g_1, \dots, g_m$ and
- a constant $c' > 0$ depending only on $g_1, \dots, g_m$ and f

such that

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Use rounding procedures in general polynomial optimization?

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Bad Corollary: Suppose $(*)$ holds, $n \geq 2$ and S has nonempty interior. Then there is k such that there is no ℓ such that for all $f \in V_k$, ${P_\ell}^* = P^*$.

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