Arbeitsgemeinschaft on The Kadison-Singer Conjecture

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Univariate stable polynomials

In this talk we tried to give a very basic introduction to univariate stable polynomials and to illustrate their usefulness in combinatorics. More complete introductions into this subject are [7] and [8].

All polynomials will be real or complex polynomials in one variable X. We denote the corresponding algebra of polynomials by $\mathbb{R}[X]$ and $\mathbb{C}[X]$, respectively.

1. Stable polynomials

We call $f \in \mathbb{C}[X]$ *stable* if f is the zero polynomial or f has no complex roots with positive imaginary part, i.e., $f = 0$ or $\forall z \in \mathbb{C} : (\operatorname{Im}(z) > 0 \implies f(z) \neq 0)$.

There are many other notions of stability with the upper half plane replaced by other regions. The term "stable" is motivated by control theory where the stable behavior of a system can often be related to stability (in one sense or the other) of a polynomial.

The Gauß-Lucas Theorem [3, Theorem 2.1.1] says that for a non-constant complex polynomial the zeros of its derivative are convex combinations of its zeros:

$$\forall p \in \mathbb{C}[X] \setminus \mathbb{C} : \{z \in \mathbb{C} \mid p'(z) = 0\} \subseteq \operatorname{conv}\{z \in \mathbb{C} \mid p(z) = 0\}.$$

As a corollary, derivatives of stable polynomials are stable.

2. Real stable polynomials

Stable polynomials in $\mathbb{R}[X]$ are called *real stable* (also *real-rooted* and sometimes *hyperbolic* or *real zero polynomial*). These are the polynomials of the form

$$\lambda(X - a_1)\cdots(X - a_n) \qquad (n \in \mathbb{N}_0, \lambda, a_1, \dots, a_n \in \mathbb{R}).$$

To see that derivatives of real stable polynomials are real stable, one can simply use Rolle's theorem instead of the Gauß-Lucas Theorem.

If $p = (X - a_1)\cdots(X - a_n)$ with $\lambda_i \in \mathbb{C}$, then its reciprocal $p^* := X^n p\left(\frac{1}{X}\right) = (1 - a_1X)\cdots(1 - a_nX) =: 1 - p^\sharp$ has the same coefficients in reversed order and we have a formal identity

$$\sum_{k=1}^{\infty} \frac{1}{k} (\underbrace{a_1^k + \cdots + a_n^k}_{=:N_k}) X^k = -\log p^* = -\log(1 - p^\sharp) = \sum_{k=1}^{\infty} \frac{1}{k} p^{\sharp k}$$

and therefore the k-th *Newton sum* N_k can be expressed polynomially in the coefficients of p. Hence the *Hermite-matrix*

$$H(p) := \begin{pmatrix} N_0 & N_1 & N_2 & \cdots & N_{n-1} \\ N_1 & N_2 & & & \vdots \\ N_2 & & & & \vdots \\ \vdots & & & & N_{2n-3} \\ N_{n-1} & \cdots & \cdots & N_{2n-3} & N_{2n-2} \end{pmatrix} = V(p)^T V(p)$$

can be written easily in terms of the coefficients of p in contrast to the *Vandermonde-matrix*

$$V(p) := \begin{pmatrix} 1 & a_1 & \cdots & a_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & a_n & \cdots & a_n^{n-1} \end{pmatrix}.$$

If p is stable, then $H(p)$ is clearly positive semidefinite. By a theorem of Hermite and Sylvester [4, Theorem 4.57], the converse is also true. Since the positive semidefiniteness of a real symmetric matrix can easily be decided without computing its eigenvalues, this gives a very efficient test for real stability of polynomials.

Since Descarte's rule of signs [4, Theorem 2.33] is exact for real stable polynomials [4, Corollary 2.49, Theorem 2.47], we have: If $p = (X - a_1) \cdots (X - a_n)$ with $a_i \in \mathbb{R}$, then $\#\{i \mid a_i > 0\}$ is the number of signs in the coefficient sequence of p (disregarding zero coefficients). As a special case, which is however trivial, a non-zero real stable polynomial has no positive roots if and only if it has no negative coefficients.

By Edrei's equivalence theorem [7, Theorem 4.9], a real non-zero polynomial $c_n X^n + \cdots + c_0$ ($c_i \in \mathbb{R}$) is stable without positive roots if and only if the infinite (lower triangular Toeplitz) matrix

$$\begin{pmatrix} c_0 & & & \\ c_1 & c_0 & & \\ c_2 & c_1 & c_0 & \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

is *totally positive*, i.e., all its minors are nonnegative.

3. Newton's inequalities and unimodality

A sequence $(c_0, \ldots, c_n)$ of real nonnegative numbers is called *unimodal* if there exists an $m \in \{0, \ldots, n\}$ such that $c_0 \leq \cdots \leq c_{m-1} \leq c_m \geq c_{m+1} \geq \cdots \geq c_n$. We say it has *no internal zeros* if $\{i \mid a_i \neq 0\}$ is an interval in $\{0, \ldots, n\}$. We say it is *log-concave* if it has no internal zeros and $c_{k-1} c_{k+1} \leq c_k^2$ for all $k \in \{1, \ldots, n-1\}$. One checks easily that the sequence $\left(\binom{n}{0}, \ldots, \binom{n}{n}\right)$ of binomial coefficients is log-concave. We say that $(c_0, \ldots, c_n)$ is *ultra log-concave* if it has no internal zeros and satisfies *Newton's inequalities*

$$\frac{c_{k-1}}{\binom{n}{k-1}} \frac{c_{k+1}}{\binom{n}{k+1}} \leq \left(\frac{c_k}{\binom{n}{k}} \right)^2$$

for $k \in \{1,\dots,n-1\}$. One checks easily

$$\text{“ultra log-concave} \implies \text{log-concave} \implies \text{unimodal”}.$$

Looking at 2×2 minors in Edrei's theorem, one sees that a real stable polynomial without negative coefficients has log-concave coefficient sequence. However, it has even ultra log-concave coefficient sequence: This can be seen as follows: Fix three consecutive terms in the given polynomial. Kill all terms of lower degree by taking an appropriate higher derivative. To get rid of terms of higher degree, take the reciprocal and take again a suitable higher derivative. All these operations preserve stability. Now you end up with a real stable quadratic polynomial. Its discriminant must be nonnegative and this gives exactly Newton's inequalities.

As an example of this method, consider the unsigned Stirling numbers of the first kind $\left[{n \atop k}\right] := \#\{\sigma \in S_n \mid \sigma \text{ has exactly } k \text{ cycles}\}$ $(k,n \in \mathbb{N}_0,\ 0 \le k \le n)$. Here 1-cycles, i.e., fixed points of the permutation count. We claim that $\left(\left[{n \atop 0}\right],\dots,\left[{n \atop n}\right]\right)$ is unimodal. To prove this, we show even that it is ultra log-concave. It suffices to show that $p := \sum_{k=0}^{n} \left[{n \atop k}\right] X^k$ is stable. But $p = X(X+1)\cdots(X+n-1) =: X^{(n)}$ (“rising factorial”). This can be seen easily by counting permutations whose cycles are colored (thinking of X as the number of colors) in two different ways: One way is by grouping together permutations with the same number of cycles. The other way is by successively deciding for each number between 1 and n whether it should go into a new cycle (in which case a color has to be chosen) or whether it should be inserted in one of the already existing cycles (in which case it has to be inserted at some position in these cycles).

4. Stability preservers

Borcea and Brändén characterized in 2009 all *linear stability preservers* $\mathbb{R}[X] \to \mathbb{R}[X]$ and $\mathbb{C}[X] \to \mathbb{C}[X]$ (an example of which is $p \mapsto p'$). This involves however a notion of multivariate stability and therefore goes beyond the scope of this talk [5]. Brenti [2] proved in 1989 that the restriction of the linear map $\mathbb{R}[X] \to \mathbb{R}[X], X^k \mapsto X^{(k)}$ (the “rising factorial” from above) to polynomials with only nonnegative coefficients preserves stability (this is the correct part of [7, Theorem 4.6], the other part being trivially wrong as the counterexample $(X+1)(X+2)$ shows).

As an example of how to use stability preserving maps, consider a variant of the above Stirling numbers: Define a *cycle* of a *function* $f\colon \{1,\dots,n\} \to \{1,\dots,n\}$ as a connected component of the graph $\{\{x, f(x)\} \mid x \in \{1,\dots,n\}, x \ne f(x)\}$. For $k,n \in \mathbb{N}_0$ with $0 \le k \le n$, define $\left\{{n \atop k}\right\}$ as the number of functions $\{1,\dots,n\} \to \{1,\dots,n\}$ with exactly k cycles. We claim that $\left(\left\{{n \atop 0}\right\},\dots,\left\{{n \atop n}\right\}\right)$ is unimodal. Again, we show even that it is ultra log-concave. Without loss of generality $n \ge 1$. Using the formula for the number of rooted forests on n vertices with exactly k trees from [6, end of Chapter 30] (I am grateful to Benjamin Matschke for showing me this formula and relating it to this), it is an exercise to show that $\left\{{n \atop k}\right\} =$

$\sum_{i=1}^{n}\binom{n-1}{i-1}n^{n-i}\left[{i \atop k}\right]$ for $n,k\in\mathbb{N}_0$ with $0\le k\le n$. Hence

$$p := \sum_{k=0}^{n}\left\{{n \atop k}\right\}X^k = \sum_{i=1}^{n}\binom{n-1}{i-1}n^{n-i}\underbrace{\sum_{k=0}^{n}\left[{i \atop k}\right]X^k}_{=X^{(i)}}.$$

But p is stable by Brenti's result mentioned above since $q := \sum_{i=1}^{n}\binom{n-1}{i-1}n^{n-i}X^i = X(X+n)^{n-1}$ is stable.

5. Pólya-Schur multiplier sequences

Already in 1914, Pólya and Schur characterized in their fulminant work [1] all *linear stability preservers* $\mathbb{R}[X]\to\mathbb{R}[X]$ of the form $X^k\mapsto\lambda_k X^k$ for some sequence $(\lambda_k)_{k\in\mathbb{N}}$. We mentioned the main facts of this beautiful work similarly to the exposition in [7, Section 4.3]. A detailed account of this theory can be found in [3, Sections 5.4 and 5.7]

References

[1] G. Pólya, I. Schur, *Über zwei Arten von Faktorfolgen in der Theorie der algebraischen Gleichungen*, J. Reine Angew. Math. **144** (1914) 89–113 [`http://resolver.sub.uni-goettingen.de/purl?PPN243919689`] ↑4

[2] F. Brenti, *Unimodal, log-concave and Pólya frequency sequences in combinatorics*, Mem. Amer. Math. Soc. **81** (1989), no. 413 ↑3

[3] Q. Rahman, G. Schmeisser, *Analytic theory of polynomials*, London Mathematical Society Monographs, New Series, 26, The Clarendon Press, Oxford University Press, Oxford, 2002 ↑1, 4

[4] S. Basu, R. Pollack, M.-F. Roy, *Algorithms in real algebraic geometry*, Algorithms and Computation in Mathematics, **10**, Springer-Verlag, Berlin, 2003 ↑2

[5] J. Borcea, P. Brändén, *The Lee-Yang and Pólya-Schur programs I*, Linear operators preserving stability, Invent. Math. **177** (2009), no. 3, 541–569 ↑3

[6] M. Aigner, G.M. Ziegler, *Proofs from The Book*, Fourth edition. Springer-Verlag, Berlin, 2010 ↑3

[7] R. Pemantle, *Hyperbolicity and stable polynomials in combinatorics and probability*, Current developments in mathematics, 2011, 57–123, Int. Press, Somerville, MA, 2012 ↑1, 2, 3, 4

[8] N.K. Vishnoi, *Zeros of polynomials and their applications to theory: a primer*, in: FOCS 2013 Workshop on Zeros of Polynomials and their Applications to Theory, Microsoft Research, 2013, pp. 1–18 [`http://msr-waypoint.com/en-us/um/people/nvishno/site/Publications_files/ZerosIntro.pdf`] ↑1