The iterated square root ideal[1] and the sums of squares dual of a semidefinite program[2]

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Solving Polynomial Equations
Algebraic Geometry with a view towards applications
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[1] joint work with Tim Netzer
[2] joint work with Igor Klep

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For given generators of an ideal $I \subseteq \mathbb{C}[\underline{X}]$, compute generators of its radical ideal $\sqrt{I} := \{f \in \mathbb{C}[\underline{X}] \mid \exists N \in \mathbb{N} : f^N \in I\}$.

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By Hilbert's Nullstellensatz (1893), $\sqrt{I}$ is the set of all polynomials vanishing on $V_{\mathbb{C}}(I) = \{x \in \mathbb{C}^n \mid \forall g \in I : g(x) = 0\}$.

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To see this, let $a, b \in A$ and $s, t \in \sum A^2$ such that $a^2 + s, b^2 + t \in I$.
Then $(a+b)^2 + (a-b)^2 + s + s + t + t = 2(a^2 + s) + 2(b^2 + t) \in I$.

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$$\sqrt[\mathrm{r}]{I} = \bigcup_{k \in \mathbb{N}} \sqrt[2^k]{I}$$

and

$$\sqrt[\mathrm{r}]{I} = \sqrt[2^k]{I}$$

for some $k \in \mathbb{N}$ if A is noetherian.

A semidefinite characterization of the square root ideal

For each $k \in \mathbb{N}$, let $\overrightarrow{x_k}$ denote the column vector of the first k monomials in $\mathbb{R}[\underline{X}]$ with respect to an arbitrary but fixed numbering of the monomials.

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If $I \subseteq \mathbb{R}[\underline{X}]$ is an ideal, then

$$\sqrt[2]{I} = \{\operatorname{pol}(W) \mid k, m \in \mathbb{N}, U \in S\mathbb{R}^{m \times m}, W \in \mathbb{R}^{k \times m}, \\ \operatorname{pol}(U) \in I, U \succeq W^* W\}.$$

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Note that here $U \succeq W^* W \iff \begin{pmatrix} U & W^* \\ W & I_k \end{pmatrix} \succeq 0.$

Semidefinite programming duality

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(P) & \text{minimize} & c_1 x_1 + \cdots + c_n x_n \\
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(D) & \text{maximize} & -\,tr(A_0 S) \\
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(P) & \text{minimize} & \ell(x) \\
 & \text{subject to} & x \in \mathbb{R}^n \\
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(P)	minimize	$\ell(x)$
	subject to	$x \in \mathbb{R}^n$
		$L(x) \succeq 0$

(D)	maximize	a
	subject to	$S \in S\mathbb{R}^{m \times m}, a \in \mathbb{R}$
		$S \succeq 0$
		$\ell - a = \operatorname{tr}(LS)$

We call a polynomial $\ell \in \mathbb{R}[\underline{X}]$ linear if it is of degree at most 1,

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Remark: The convex cone of sos-matrices of degree at most $2d$ is semidefinitely representable, i.e., a projection of a spectrahedron. This is just a generalization of the well known Gram matrix method for $\mathbb{R}[\underline{X}] = \mathbb{R}[\underline{X}]^{1 \times 1}$ due to Kojima and Hol & Scherer. In other words, being an sos-matrix of degree at most $2d$ can be expressed as a constraint of a semidefinite program by means of additional variables. The size of the semidefinite description (of this constraint) depends polynomially on d for fixed n.

Idea: Allow more general certificates for nonnegativity on S_L.

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A naive sos Farkas' lemma for semidefinite programming

Observation: If $L \in \mathbb{R}[\underline{X}]^{m\times m}$ is a pencil and $S \in \mathbb{R}[\underline{X}]^{m\times m}$ is an sos-matrix, then $\mathrm{tr}(LS)$ is obviously a polynomial nonnegative on S_L.

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Problems: This gives a way of expressing infeasibility of an SDP by feasibility of another SDP whose size is however exponential. Moreover this is not yet strong duality.

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(also $m = n + 1$ und $k = 1 + 2n + \binom{n}{2}$).

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Lemma: Let $\ell_1, \dots, \ell_t \in \mathbb{R}[\underline{X}]$ be linear and $q_1, \dots, q_t \in \mathbb{R}[\underline{X}]$ be quadratic. Let $U \in \mathbb{R}^{m \times m}$ be such that

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Then there exists $\lambda > 0$ and $W \in \mathbb{R}^{k\times m}$ such that $\lambda U \succeq W^* W$ and

$$\overrightarrow{y}^* W \overrightarrow{x} = \ell_1 q_1 + \cdots + \ell_t q_t.$$

The sums of squares dual of a semidefinite program

It can now be shown that the following provides a duality theory for semidefinite programming where strong duality (zero gap & dual attainment) always holds. Note that the size of the dual (which we do not explicit) is polynomial in the size of the primal.

Theorem: Let $L \in \mathbb{R}[\underline{X}]^{m\times m}$ be a pencil and $\ell \in \mathbb{R}[\underline{X}]$ be linear. Then $\ell \geq 0$ on S_L if and only if there exist

- quadratic sos-matrices $S_1, \dots, S_n \in \mathbb{R}[\underline{X}]^{m\times m}$,
- matrices $U_1, \dots, U_n \in S\mathbb{R}^{m\times m}$, $W_1, \dots, W_n \in \mathbb{R}^{k\times m}$, $S \in S\mathbb{R}^{m\times m}_{\succeq 0}$ and
- a real number $a \geq 0$

such that

$$\begin{aligned}
&\overrightarrow{x}^* U_i \overrightarrow{x} + \overrightarrow{y}^* W_{i-1} \overrightarrow{x} + \operatorname{tr}(L S_i) = 0 && (i \in \{1, \dots, n\}),\\
&U_i \succeq W_i^* W_i && (i \in \{1, \dots, n\}),\\
&\ell + \overrightarrow{y}^* W_n \overrightarrow{x} = a + \operatorname{tr}(LS)
\end{aligned}$$

where $W_0 := 0$.

Based on other ideas, such a duality theory has also been given by Matt Ramana:

M. Ramana: An exact duality theory for semidefinite programming and its complexity implications
Math. Programming **77** (1997), no. 2, Ser. B, 129–162
http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.47.8540&rep=rep1&type=pdf
http://dx.doi.org/10.1007/BF02614433

See also:

Ramana & Tunçel & Wolkowicz: Strong duality for semidefinite programming
SIAM J. Optim. **7** (1997), Issue 3, 641–662 (1997)
http://www.math.uwaterloo.ca/~ltuncel/publications/strong-duality.pdf
http://dx.doi.org/10.1137/S1052623495288350