ITERATED RINGS OF BOUNDED ELEMENTS: ERRATUM

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Abstract. We close a gap in the author's thesis [S1, S2].

The author's proof of [S1, S2, Lemma 4.10] **is not correct. In this note, we show that this does not affect the validity of any other statement in** [S1, S2]**.** We will observe that the lemma in question holds in any of the following important special cases:

(a) $T = \sum A^2$
(b) A contains a f.g. subalgebra C such that T is as a preordering generated by $T \cap C$.
(c) T is as a preordering finitely generated (this is just a special case of (b)).
(d) A is a reduced ring.

Unfortunately, we don't know whether the lemma holds without any such additional hypothesis.

1. Acknowledgements

The author would like to thank Claus Scheiderer and Mark Olschok who discovered the gap in a research seminar in Duisburg and reported it to us. The author was able to close the gap, but it was again Claus Scheiderer who came up with a more conceptual proof which we will present in Section 4 below.

2. The error

Recall the situation in the proof of the lemma. We have an extension $B \subseteq A$ of preordered rings, i.e., an extension $B \subseteq A$ of rings such that A is equipped with a preordering T and B with $T \cap B$. We have $\mathfrak{p} \in \operatorname{Spec} A$ and $\mathfrak{q} = \mathfrak{p} \cap B$. Then $B/\mathfrak{q}$ can be viewed as a subring of $A/\mathfrak{p}$ but perhaps **not** as a **preordered** subring, contrary to what is said in [S1, S2]. Here $A/\mathfrak{p}$ and $B/\mathfrak{q}$ are equipped with the preorderings

$$T_{A/\mathfrak{p}} = \{t + \mathfrak{p} \mid t \in T\} \qquad \text{and} \qquad T_{B/\mathfrak{q}} = \{t + \mathfrak{q} \mid t \in T \cap B\},$$

respectively. The problem is that for some $b \in B$ it could happen that $b - t$ lies in $\mathfrak{p}$ for some $t \in T$ but not in $\mathfrak{q}$ for any $t \in T$. Consequently, it is not guaranteed whether the preorderings on the quotient field $\operatorname{qf}(A/\mathfrak{p}) = \operatorname{qf}(B/\mathfrak{q})$ induced (or generated) by $T_{A/\mathfrak{p}}$ and $T_{B/\mathfrak{q}}$ coincide. Hence it is not clear whether $T \subseteq P$ when P is chosen like in the proof under review.

3. Cases where the proof still works

In case (a), $T \subseteq P$ holds trivially. In case (b), we may assume that $C \subseteq B$. This implies $T \cap C \subseteq T \cap B \subseteq Q \subseteq P$ and therefore $T \subseteq P$.

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4. Proof of the lemma for reduced rings

In this section, we will prove that [S1, S2, Lemma 4.10] holds under the additional assumption (d) that A is a *reduced* ring, i.e., contains no nonzero nilpotent elements. This is done in Lemma 7 below.

We need some well-known facts from commutative ring theory whose proofs we include for the convenience of the reader. Let A always denote a commutative ring (with unity, of course).

Lemma 1. *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be prime ideals of a commutative ring A satisfying $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for $i \neq j$. Then $\mathfrak{p}_i \not\subseteq \bigcup_{j \neq i} \mathfrak{p}_j$ for all $i \in \{1, \dots, n\}$.*

Proof. For every (i, j) with $j \neq i$, choose $a_{ij} \in \mathfrak{p}_i \setminus \mathfrak{p}_j$. Then we have for instance

$$\sum_{i=2}^{n} \prod_{j \neq i} a_{ij} \in \mathfrak{p}_1 \setminus \bigcup_{j \neq i} \mathfrak{p}_j.$$

□

Lemma 2. *Let A be a reduced ring with only finitely many pairwise distinct minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$. Then the zero divisors in A (i.e., the elements $a \in A$ for which there is some $0 \neq b \in A$ with $ab = 0$) are exactly the elements of $\mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_n$.*

Proof. Consider an element of $\mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_n$, say $a \in \mathfrak{p}_1$. By the preceding lemma, we can choose $b_i \in \mathfrak{p}_i \setminus \bigcup_{j \neq i} \mathfrak{p}_j$ for $i \in \{2, \dots, n\}$. Then $ab_2 \cdots b_n \in \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_n = 0$ but $b_2 \cdots b_n \notin \mathfrak{p}_1$, in particular $b_2 \cdots b_n \neq 0$. Thus a is a zero divisor.

Conversely, suppose $a \in A \setminus (\mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_n)$. We show that a is not a zero divisor. Suppose therefore that $b \in A$ and $ab = 0$. From $ab = 0 \in \mathfrak{p}_i$ and $a \notin \mathfrak{p}_i$ it follows that $b \in \mathfrak{p}_i$ for all $i \in \{1, \dots, n\}$. Hence b lies in $\mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_n = 0$. □

Definition 3. The *total quotient ring* of A is the ring

$$\mathrm{Quot}(A) := S^{-1}A$$

where S is the set non zero divisors of A (note that $1 \in S$ and $SS \subseteq S$).

Lemma 4. *The canonical homomorphism $A \to \mathrm{Quot}(A)$ is an embedding.*

Proof. Suppose $a \in A$ and $a/1 = 0 \in \mathrm{Quot}(A)$. Then there is some non zero divisor s of A such that $as = 0$ in A. But then, of course, $a = 0$. □

Lemma 5. *Let A be reduced with only finitely many pairwise distinct minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$. Then there is a canonical isomorphism*

$$\mathrm{Quot}(A) \xrightarrow{\cong} \mathrm{qf}(A/\mathfrak{p}_1) \times \cdots \times \mathrm{qf}(A/\mathfrak{p}_n).$$

Proof. Let S denote the set of non zero divisors of A. Then $\mathrm{Quot}(A) = S^{-1}A$ by Definition 3. The prime ideals of $S^{-1}A$ correspond to the prime ideals of A contained in $A \setminus S = \mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_n$ (Lemma 2). But by Lemma 1, the only prime ideals of A contained in $\mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_n$ are $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ themselves. Therefore $S^{-1}\mathfrak{p}_1, \dots, S^{-1}\mathfrak{p}_n$ are (all) pairwise distinct maximal ideals of $S^{-1}A$. In particular, these ideals are pairwise coprime. Moreover it is easy to see that

$$S^{-1}\mathfrak{p}_1 \cap \ldots \cap S^{-1}\mathfrak{p}_n = 0$$

and that there is a canonical isomorphism

$$S^{-1}A/S^{-1}\mathfrak{p}_i \xrightarrow{\cong} \mathrm{qf}(A/\mathfrak{p}_i)$$

for all $i \in \{1, \dots, n\}$. Our claim therefore follows by applying the Chinese Remainder Theorem to the ring $S^{-1}A$. □

Lemma 6. *Let A be reduced with only finitely many pairwise distinct minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$. Let B be a subring of A such that the following conditions hold:*

(1) The canonical embedding $\mathrm{qf}(B/(\mathfrak{p}_i \cap B)) \hookrightarrow \mathrm{qf}(A/\mathfrak{p}_i)$ is a field isomorphism for all i.

(2) $\mathfrak{p}_i \cap B \not\subseteq \mathfrak{p}_j \cap B$ for all $i \neq j$.

Then there is a canonical isomorphism $\mathrm{Quot}(B) \xrightarrow{\cong} \mathrm{Quot}(A)$.

Proof. Every minimal prime ideal of B is of the form $\mathfrak{p}_i \cap B$ for some $i \in \{1, \dots, n\}$ (confer [S1, S2, Remark 4.8]). Conversely, we argue that every $\mathfrak{p}_i \cap B$ is actually a minimal prime ideal of B. To see this, observe that $\mathfrak{p}_i \cap B$ contains in any case a minimal prime ideal. Hence $\mathfrak{p}_j \cap B \subseteq \mathfrak{p}_i \cap B$ for some minimal prime ideal $\mathfrak{p}_j \cap B$ of B. Condition (2) forces $i = j$ showing that $\mathfrak{p}_i \cap B$ is itself a minimal prime ideal. Our claim follows now from condition (1), the preceding lemma and the fact that $\mathfrak{p}_1 \cap B, \dots, \mathfrak{p}_n \cap B$ are exactly the *pairwise distinct* minimal prime ideals of B. □

Lemma 7. *Suppose A is f.f., almost archimedian and reduced. Then*

$$A = \bigcup_{\rightarrow} B, \quad \textit{where } B \textit{ ranges over f.g., almost archimedian algebras.}$$

Proof. Denote the finitely many minimal pairwise distinct prime ideals of A by $\mathfrak{p}_1, \dots, \mathfrak{p}_n$. Since A is f.f., we can choose $a_1, \dots, a_m \in A$ such that $\mathrm{qf}(A/\mathfrak{p}_i)$ is for each $i \in \{1, \dots, n\}$ generated by $a_1 + \mathfrak{p}_i, \dots, a_m + \mathfrak{p}_i$ as a field over K. Choose moreover $b_{ij} \in \mathfrak{p}_i \setminus \mathfrak{p}_j$ for all $i \neq j$. Clearly, $A = \bigcup_{\rightarrow} B$ where B ranges over all f.g. subalgebras of A containing all of the finitely many a_i and b_{ij}.

Now we fix such a B. It remains to show that B is almost archimedean. Fix an arbitrary $Q \in \mathrm{Sper}\, B$ such that $Q \cap -Q$ is a minimal prime ideal of B. We have to show that Q is archimedean. There is i such that $Q \cap -Q = \mathfrak{p}_i \cap B$ by [S1, S2, Remark 4.8]. Since B contains all a_i and b_{ij}, conditions (1) and (2) from Lemma 6 are satisfied. From (1) we see that $\mathrm{qf}(A/\mathfrak{p}_i) = \mathrm{qf}(B/(\mathfrak{p}_i \cap B))$ as fields (not necessarily as preordered fields!). Consequently, there is some ordering P of the ring A such that $Q = P \cap B$ and $P \cap -P = \mathfrak{p}_i$. It remains to show that $P \in \mathrm{Sper}\, A$ or, in more explicit words, $T \subseteq P$. Once we have shown this, it follows that P is archimedean since A is almost archimedean. But then Q must of course be archimedean, too.

To show $T \subseteq P$, we use Lemma 5 saying that there is a canonical isomorphism $\mathrm{Quot}(B) \xrightarrow{\cong} \mathrm{Quot}(A)$. Consider an arbitrary $t \in T$. Since $t/1$ lies in the image of this isomorphism, there is some $b \in B$ and some non zero divisor s in B such that $b/s = t/1$ holds in $\mathrm{Quot}(A)$. This implies $b/1 = st/1$ in $\mathrm{Quot}(A)$ and a fortiori $b = st$ in A by Lemma 4. Hence $s^2 t \in T \cap B \subseteq Q \subseteq P$. If s were an element of $P \cap -P = \mathfrak{p}_i$, then it would lie in the minimal prime ideal $\mathfrak{p}_i \cap B = Q \cap -Q$ of B which is impossible by Lemma 2. From $s \notin P \cap -P$ it follows now that $t \in T$. □

5. Closing the gap

Finally, we show that Lemma 7 is enough to ensure the validity of all results of [S1, S2], with the only possible exception of [S1, S2, Lemma 4.10], of course. In fact, [S1, S2, Lemma 4.10] is only applied once in [S1, S2], namely in the proof of [S1, S2, Theorem 4.13] (and its variant for quadratic modules and semiorderings in [S1, S2, Subsection 6.2] which can be treated completely analogously).

Hence it suffices to show that we can restrict ourselves in the proof of [S1, S2, Theorem 4.13] to *reduced* A since then we can apply Lemma 7 instead of [S1, S2, Lemma 4.10]. Assume therefore that [S1, S2, Theorem 4.13] has already been shown for reduced A.

Now for general A, denote the nilradical of A by $\operatorname{Nil}(A)$. Suppose $A = H(A)$. Then $A/\operatorname{Nil}(A) = H(A/\operatorname{Nil}(A))$ and therefore $A/\operatorname{Nil}(A) = H'(A/\operatorname{Nil}(A))$ since $A/\operatorname{Nil}(A)$ is reduced. Suppose $a \in A$. We show that $a \in H'(A)$. From $a + \operatorname{Nil}(A) \in A/\operatorname{Nil}(A) = H'(A/\operatorname{Nil}(A))$ we obtain $\nu \in \mathbb{N}$ and $b \in \operatorname{Nil}(A)$ such that $\nu - a + b \in T$. Clearly $b \in H'(A)$ by [S1, S2, Lemma 4.1]. This supplies us with a $\nu' \in \mathbb{N}$ such that $\nu' - b \in T$. Finally, we see that

$$(\nu + \nu') - a = (\nu - a + b) + (\nu' - b) \in T + T \subseteq T.$$

Since $a \in A$ was arbitrary, we see that $A = H'(A)$ as desired.

References

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