Model Theory of Transseries

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Overview

I. Transseries

II. Some Conjectures about Transseries

III. Recent Results

(joint with Lou van den Dries and Joris van der Hoeven)

I. Transseries

A reminder on Laurent series

The field $\mathbb{R}((x^{-1}))$ of (formal) Laurent series over $\mathbb{R}$ in *descending* powers of x consists of all series

$$f(x) = \underbrace{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x}_{\text{infinite part of } f} + a_0 + \underbrace{a_{-1} x^{-1} + a_{-2} x^{-2} + \cdots}_{\text{infinitesimal part of } f}$$

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Exponentiation for elements of $\mathbb{R}[[x^{-1}]]$ can be defined:

$$\begin{aligned}
&\exp(a_0 + a_{-1}x^{-1} + a_{-2}x^{-2} + \cdots) \\
&= \mathrm{e}^{a_0} \sum_{n=0}^{\infty} \frac{1}{n!}(a_{-1}x^{-1} + a_{-2}x^{-2} + \cdots)^n \\
&= \mathrm{e}^{a_0}(1 + b_1 x^{-1} + b_2 x^{-2} + \cdots) \quad \text{for suitable } b_1, b_2, \ldots \in \mathbb{R}.
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- x^{-1} has no antiderivative in $\mathbb{R}((x^{-1}))$.
- $\mathbb{R}((x^{-1}))$, as a *differential* field, existentially defines $\mathbb{Z}$.

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$$\mathrm{e}^{\mathrm{e}^{x}+\mathrm{e}^{x/2}+\mathrm{e}^{x/4}+\cdots}-3\mathrm{e}^{x^2}+5x^{\sqrt{2}}-(\log x)^{\pi}+1+x^{-1}+x^{-2}+\cdots+\mathrm{e}^{-x}.$$

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The field $\mathbb{T}$ has a somewhat lengthy inductive definition, a feature of which is that series like

$$\frac{1}{x}+\frac{1}{e^{x}}+\frac{1}{e^{e^{x}}}+\frac{1}{e^{e^{e^{x}}}}+\cdots, \qquad \frac{1}{x}+\frac{1}{x\log x}+\frac{1}{x\log x\log\log x}+\cdots$$

are excluded. (“$\mathbb{T}$ is not spherically complete.”)

Working in $\mathbb{T}$

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With this ordering, $\mathbb{T}$ becomes an *ordered field* with

$$\mathbb{R} < \cdots < \log\log x < \log x < x < e^x < e^{e^x} < \cdots .$$

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$$\begin{aligned}
\sinh &:= \tfrac{1}{2}\mathrm{e}^{x} - \tfrac{1}{2}\mathrm{e}^{-x} \in \mathbb{T}^{>0} \\
\exp(\sinh) &= \exp\left(\tfrac{1}{2}\mathrm{e}^{x}\right) \cdot \exp\left(-\tfrac{1}{2}\mathrm{e}^{-x}\right) \\
&= \mathrm{e}^{\frac{1}{2}\mathrm{e}^{x}} \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\tfrac{1}{2}\mathrm{e}^{-x}\right)^{n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! 2^n} \mathrm{e}^{\frac{1}{2}\mathrm{e}^{x} - nx}, \\
\log(\sinh) &= \log\left(\frac{\mathrm{e}^{x}}{2}\left(1 - \mathrm{e}^{-2x}\right)\right) = x - \log 2 - \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{e}^{-2nx}.
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The structure $(\mathbb{T}, 0, 1, +, \cdot, \leqslant, \exp)$ is well understood:

$$(\mathbb{R}, \ldots, \exp) \preccurlyeq (\mathbb{T}, \ldots, \exp).$$

(MACINTYRE-MARKER-VAN DEN DRIES, 1990s)

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We obtain a *derivation* $f \mapsto f' : \mathbb{T} \to \mathbb{T}$ on the field $\mathbb{T}$:

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- The *dominance relation* on $\mathbb{T}$: for $0 \neq f, g \in \mathbb{T}$,

$$f \preccurlyeq g \quad :\Longleftrightarrow \quad \begin{cases} \text{(leading monomial of } f) \leqslant \\ \text{(leading monomial of } g). \end{cases}$$

  So for example

$$\mathrm{e}^{-x-x^{1/2}-x^{1/4}-\cdots} \prec -5\mathrm{e}^{-x/2} - \mathrm{e}^{-x}.$$

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Origins and applications of transseries

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- for example, functions definable in many (all?) exponentially bounded o-minimal expansions of the real field (like the ordered exponential field $\mathbb{R}$).

No function has presented itself in analysis the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmic-exponential terms.
(G. H. HARDY, Orders of Infinity, 1910.)

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VAN DER HOEVEN shows that the differential subfield $\mathbb{T}^{\mathrm{da}}$ of $\mathbb{T}$ consisting of the differentially algebraic transseries has an analytic counterpart.

Transserties with analytic meaning

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This can be made precise using the language of model theory.

II. Some Conjectures about Transseries

The $\mathbb{T}$-Conjecture

From now on, we view $\mathbb{T}$ as a (model-theoretic) structure where we single out the primitives

$$0,\ 1,\ +,\ \cdot,\ \partial \text{ (derivation)},\ \leqslant \text{ (ordering)},\ \preccurlyeq \text{ (dominance)}.$$

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$\mathbb{T}$ is model complete.

(The inclusion of $\preccurlyeq$ is necessary.)

This can be expressed geometrically in terms of systems of algebraic differential (in)equations. (Similar to GABRIELOV's "theorem of the complement" for real subanalytic sets.)

Define a d-**algebraic** set in $\mathbb{T}^n$ to be a zero set

$$\{y \in \mathbb{T}^n : P_1(y) = \cdots = P_m(y) = 0\}$$

of some d-polynomials

$$P_i(Y_1, \ldots, Y_n) \;=\; p_i\big(Y_1, \ldots, Y_n,\ Y_1', \ldots, Y_n',\ Y_1'', \ldots, Y_n'',\ \ldots\big)$$

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over $\mathbb{T}$. Define an *H*-**algebraic** set in $\mathbb{T}^n$ to be the intersection of a d-algebraic set in $\mathbb{T}^n$ with a set of the form

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Model completeness of $\mathbb{T}$ means (almost): the *complement* of any sub-*H*-algebraic set in $\mathbb{T}^m$ is again sub-*H*-algebraic.

Some related conjectures

1. $\mathbb{T}$ is *o-minimal at* $+\infty$: if $X \subseteq \mathbb{T}$ is sub-H-algebraic, then there is some $f \in \mathbb{T}$ with $(f, +\infty) \subseteq X$ or $(f, +\infty) \cap X = \emptyset$.
2. All sub-H-algebraic subsets of $\mathbb{R}^n \subseteq \mathbb{T}^n$ are *semialgebraic.*
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An instance of 1: if P is a one-variable d-polynomial over $\mathbb{T}$, then there is some $f \in \mathbb{T}$ and $\sigma \in \{\pm 1\}$ with sign $P(y) = \sigma$ for all $y > f$. (Related to old theorems of BOREL, HARDY, ...)

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An illustration of 2: the set of $(c_0, \dots, c_n) \in \mathbb{R}^{n+1}$ such that

$$c_0 y + c_1 y' + \cdots + c_n y^{(n)} = 0, \qquad 0 \neq y \prec 1$$

has a solution in $\mathbb{T}$ is a semialgebraic subset of $\mathbb{R}^{n+1}$.

A (slightly misleading) sample use of ❸:

Let $Y = (Y_1, \dots, Y_n)$ be a tuple of distinct d-indeterminates.

Call an m-tuple $\sigma = (\sigma_1, \dots, \sigma_m)$ of elements of $\{\preccurlyeq, \succ\}$ an *asymptotic condition,* and say that d-polynomials $P_1, \dots, P_m$ in Y over $\mathbb{T}$ *realize* σ if there is some $a \in \mathbb{T}^n$ such that

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Fix $d, n, r \in \mathbb{N}$. Then the number of asymptotic conditions $\sigma \in \{\preccurlyeq, \succ\}^m$ which can be realized by some d-polynomials $P_1, \ldots, P_m$ in Y over $\mathbb{T}$ of degree at most d and order at most r *grows only polynomially* with m.

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For this we introduce the class of *H*-fields (*H*: HARDY, HAUSDORFF, HAHN, BOREL), defined to share some basic properties with $\mathbb{T}$.

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Just like $\mathbb{T}$, such a K comes with a dominance relation:

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We also use:

$$f \asymp g \quad :\Longleftrightarrow \quad f \preccurlyeq g \;\&\; g \preccurlyeq f$$

$$f \prec g \quad :\Longleftrightarrow \quad f \preccurlyeq g \;\&\; g \not\preccurlyeq f$$

$$\Longleftrightarrow \quad \forall c \in C^{>0} : |f| \leqslant c|g| \qquad \text{“}g \text{ strictly dominates } f\text{”}$$

$$f \sim g \quad :\Longleftrightarrow \quad f - g \prec g \qquad \text{“asymptotic equivalence”}$$

Definition

We call K an H**-field** provided that

(H1) $f \succ 1 \ \Rightarrow\ f^{\dagger} > 0$;

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Examples

Every ordered differential subfield $K \supseteq \mathbb{R}$ of $\mathbb{T}$ is an *H*-field. (For example, $K = \mathbb{R}(\!(x^{-1})\!)$.)

H-fields are part of the (more flexible) category of "differential-valued fields" of ROSENLICHT (1980s).

$\mathbb{T}$-Conjecture (more precise version)

Th($\mathbb{T}$) is the model companion of the theory of *H*-fields:

$\mathbb{T}$-Conjecture + "*H*-fields are exactly the ordered differential fields embeddable into ultrapowers of $\mathbb{T}$."

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This suggests an approach to a proof:

Study the extension theory of H-fields.

Encouraged by some initial positive results, in 1998 VAN DEN DRIES and myself, later ($\sim$2000) joined by VAN DER HOEVEN, embarked on carrying out this program, which we brought to a successful conclusion last year.

Besides being a real closed H-field, $\mathbb{T}$ is Liouville closed:

We call a real closed H-field K **Liouville closed** if

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What can go wrong when forming Liouville closures may be seen from the *asymptotic couple* of K.

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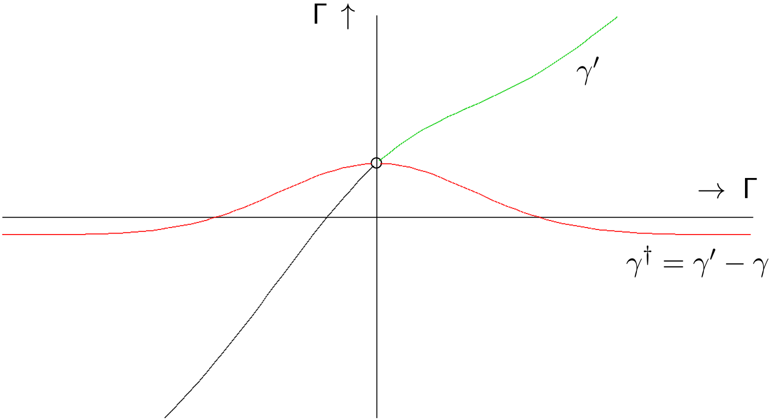
Example

For $K = \mathbb{T}$: $\quad (\Gamma, +, \leqslant) \cong (\text{group of transmonomials}, \cdot\,, \succcurlyeq)$.

The derivation ∂ of K induces a map

$$\gamma = vg \mapsto \gamma' = v(g'): \quad \Gamma^{\neq} := \Gamma \setminus \{0\} \to \Gamma.$$



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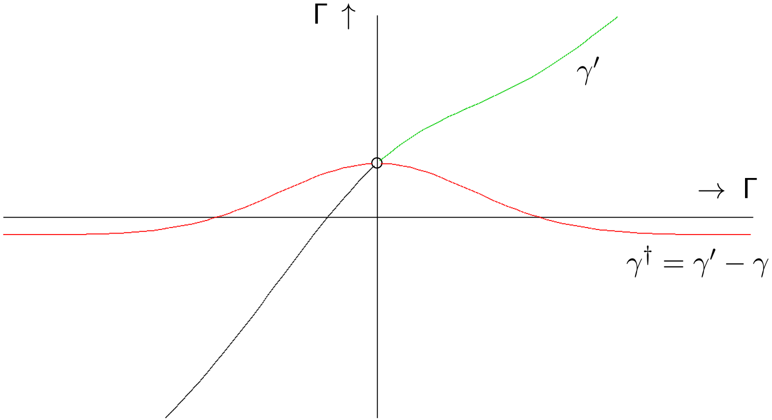

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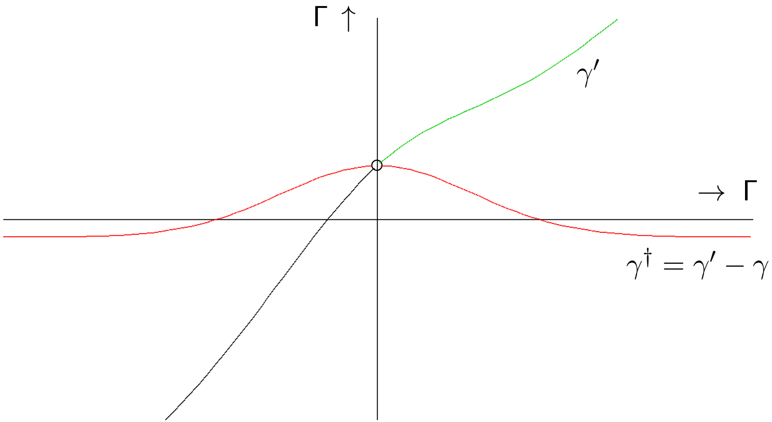

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Examples

1. $K = C$;
2. $K = \mathbb{R}((x^{-1}))$;
3. $K = \mathbb{T}$ (or any other Liouville closed K).

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In 1 we have *two* Liouville closures: if $\gamma = vg$, then we have a choice when adjoining $\int g$: make it $\succ 1$ or $\prec 1$.

In 2 we have *one* Liouville closure: if $vg = \max(\Gamma^{\neq})^{\dagger}$, then $\int g \succ 1$ in each Liouville closure of K.

In 3 we may have *one or two* Liouville closures.

III. Recent Results

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Main Theorem

The following statements axiomatize a complete theory: K is

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Corollary

$\mathbb{T}$ *is decidable; in particular:* there is an algorithm which, given d-polynomials $P_1, \dots, P_m$ in $Y_1, \dots, Y_m$ over $\mathbb{Z}[x]$, decides whether $P_1(y) = \cdots = P_m(y) = 0$ for some $y \in \mathbb{T}^n$.

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$\mathbb{T}$ *has quantifier elimination,* after also introducing primitives for multiplicative inversion and the predicates Λ, Ω, interpreted as follows, with $\ell_0 = x$, $\ell_{n+1} = \log \ell_n$:

$$\Lambda(f) \iff f < \lambda_n := \frac{1}{\ell_0} + \frac{1}{\ell_0 \ell_1} + \cdots + \frac{1}{\ell_0 \ell_1 \cdots \ell_n}, \text{ for some } n$$

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Remarks

- $\omega_n = \omega(\lambda_n)$ where $\omega(z) := -2z' - z^2$ (related to the Schwarzian derivative);
- (ω_n) also appears in classical non-oscillation theorems for 2nd order linear differential equations.

(ω_n) has no "pseudolimit" in $\mathbb{T}$: there are no $f \in \mathbb{T}$ with

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An H-field K with asymptotic integration is **ω-free** if

$$\forall f\, \exists g \left[1 \prec g \;\&\; f - \omega(-g^{\dagger\dagger}) \succcurlyeq (g^\dagger)^2\right] \quad \text{(here } a^\dagger := a'/a \text{ for } a \neq 0\text{).}$$

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Caveat: there are Liouville closed H-fields which are not ω-free!

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For example,

$$Y^{\phi} = Y, \qquad (Y')^{\phi} = \phi Y', \qquad (Y'')^{\phi} = \phi^2 Y'' + \phi' Y', \qquad \ldots$$

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The operation $P \mapsto P^\phi$ on d-polynomials can be studied using Lie-theoretic methods.

Theorem ($\sim$ 2009)

Suppose K is ω-free and $P \neq 0$. Then there exists a nonzero $N_P \in C[Y](Y')^{\mathbb{N}}$ so that for all sufficiently small admissible ϕ:

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The *newtonian* condition makes it possible to develop a *Newton diagram* method for d-polynomials.

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Some basic facts that go into the proof of our main theorem:

- Any real closed ω-free H-field has a unique *newtonization.*
- Any ω-free H-field has a unique *Newton-Liouville closure.*
- No ω-free newtonian Liouville closed H-field has a proper d-algebraic H-field extension with the same constant field.

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Corollary

$\mathbb{T}^{\mathrm{da}} = \big($*Newton-Liouville closure of* $\mathbb{R}(\ell_0, \ell_1, \dots)\big) \preccurlyeq \mathbb{T}$.

What’s next?

…see Lou’s talk.