Integration on Nash manifolds over real closed fields and Stokes' theorem

Tobias Kaiser

Universität Passau

24. Juli 2015

1. Motivation

There is a good **differential calculus** in the semialgebraic setting on arbitrary real closed fields. Classical results as

- mean value property,
- inverse function theorem,
- implicit function theorem

hold.

Reason: Differentiation is a first order property.

This is only half of the analysis. So far we do not have a good **integration theory** on arbitrary (non-archimedean) real closed fields.

Reason: Integration on the reals is not a first order property.

Example: $\int_1^x dt/t = \log(x)$ is not semialgebraic.

We have developed a full Lebesgue measure and integration theory for the category of semialgebraic sets and functions on arbitrary real closed fields such that the main properties of the real Lebesgue measure and integration hold (after the usual adjustments):

The Lebesgue measure for semialgebraic sets on a given real closed field is

- finitely additive,
- monotone,
- translation invariant

and

- reflects elementary geometry.

The Lebesgue integral for semialgebraic functions on a given real closed field is

- linear, monotone

and

- the transformation formula,
- Lebesgue's theorem on dominated convergence,
- fundamental theorem of calculus

hold.

Note: The Lebesgue measure and integration theory cannot in general be performed inside a given real closed field, the integrals take necessarily values outside the given field. This can be seen by looking at the function $x \mapsto \int_1^x dt/t$ for $x > 0$. A very basic version of the transformation formula gives that this function defines a logarithm. But on real closed fields there are in general no reasonable logarithms with values in the field.

2. Starting point

The starting point is the work of Comte, Lion and Rolin on integration of globally subanalytic sets and functions:

Theorem: Let $n \in \mathbb{N}$ and let $p \in \mathbb{N}_0$.

(A) Let $A \subset \mathbb{R}^{p+n}$ be globally subanalytic. The following holds:

(1) The set

$$\mathrm{Fin}(A) := \{t \in \mathbb{R}^p \mid \lambda_n(A_t) < +\infty\}$$

is globally subanalytic.

(2) There is $r \in \mathbb{N}$, a real polynomial P in $2r$ variables and globally subanalytic functions $\varphi_1, \ldots, \varphi_r : \mathrm{Fin}(A) \to \mathbb{R}_{>0}$ such that

$$\lambda_n(A_t) = P\big(\varphi_1(t), \ldots, \varphi_r(t), \log(\varphi_1(t)), \ldots, \log(\varphi_r(t))\big)$$

for all $t \in \mathrm{Fin}(A)$.

(B) Let $f:\mathbb{R}^{p+n}\to\mathbb{R}, (t,x)\mapsto f(t,x)=f_t(x)$, be globally subanalytic. The following holds:

(1) The set

$$\mathrm{Fin}(f) := \Big\{t\in\mathbb{R}^p \mid \int_{\mathbb{R}^n} |f_t(x)|\,d\lambda_n(x) < +\infty\Big\}$$

is globally subanalytic.

(2) There is $r\in\mathbb{N}$, a real polynomial P in $2r$ variables and globally subanalytic functions $\varphi_1,\ldots,\varphi_r:\mathrm{Fin}(f)\to\mathbb{R}_{>0}$ such that

$$\int_{\mathbb{R}^n} f_t(x)\,d\lambda_n(x) = P\big(\varphi_1(t),\ldots,\varphi_r(t),\log(\varphi_1(t)),\ldots,\log(\varphi_r(t))\big)$$

for all $t\in\mathrm{Fin}(f)$.

Here λ_n denotes the usual Lebesgue measure on $\mathbb{R}^n$.

Remark: If A (resp. f) is semialgebraic then $\mathrm{Fin}(A)$ (resp. $\mathrm{Fin}(f)$) is semialgebraic.

O-minimality: Let $\mathbb{R}_{\mathrm{an}}$ be the o-minimal structure generated by the restricted analytic functions. The sets (functions) definable in $\mathbb{R}_{\mathrm{an}}$ are precisely the globally subanalytic sets (functions). These are the sets that are subanalytic in the ambient projective space. The o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$ is the o-minimal structure obtained by expanding $\mathbb{R}_{\mathrm{an}}$ by the global exponential function.

Corollary: Let $n \in \mathbb{N}$ and $p \in \mathbb{N}_0$.

(A) Let $A \subset \mathbb{R}^{p+n}$ be semialgebraic. The function

$$\mathrm{Fin}(A) \to \mathbb{R}_{\geq 0}, t \mapsto \lambda_n(A_t),$$

is definable in $\mathbb{R}_{\mathrm{an,exp}}$.

(B) Let $f : \mathbb{R}^{p+n} \to \mathbb{R}$ be semialgebraic. The function

$$\mathrm{Fin}(f) \to \mathbb{R}, t \mapsto \int_{\mathbb{R}^n} f_t(x)\, d\lambda_n(x),$$

is definable in $\mathbb{R}_{\mathrm{an,exp}}$.

3. Idea of the construction

Let R be a real closed field and let A be a semialgebraic subset of some R^n. Then:

- Express A by a formula in the language of ordered rings augmented by symbols for the elements of R.
- Replace the tuple a of elements from R involved in this formula by variables to obtain a parameterized semialgebraic family $\widehat{A}$ of subsets of $\mathbb{R}^n$.
- Apply the result of Comte, Lion and Rolin and obtain the corresponding $\widehat{F} : \mathrm{Fin}(\widehat{A}) \longrightarrow \mathbb{R}_{\geq 0}$.
- If $a \notin \mathrm{Fin}(\widehat{A})$ then declare the Lebesgue measure of A as $+\infty$. Otherwise, plug in the above tuple a into $\widehat{F}$ and declare this as the Lebesgue measure of the given set A.

How to carry out the last step?

We need a suitable extension of R where this can be done in a reasonable way!

One could think of a suitable ultra power of $\mathbb{R}$ (= nonstandard extension of $\mathbb{R}$) containing R. Here we would enter the nonstandard measure and integration theory. But by this abstract choice we would loose control about the values obtained when measuring and integrating.

We use the work of Van den Dries, Macintyre and Marker.

4. Preparations

4.1. Standard real valuation:

Let R be a real closed field. Consider

- the valuation ring

$$\mathcal{O}_R := \{f \in R \mid -n \leq f \leq n \text{ for some } n \in \mathbb{N}\}$$

  (the set of **bounded** elements of R),

- the maximal ideal

$$\mathfrak{m}_R = \{f \in R \mid -\frac{1}{n} < f < \frac{1}{n} \text{ for all } n \in \mathbb{N}\}$$

  (the set of **infinitesimal** elements of R),

- the standard real valuation $v_R : R^* \to R^*/\mathcal{O}_R^*$ with value group $\Gamma_R := R^*/\mathcal{O}_R^*$,
- the residue field $\kappa_R := \mathcal{O}_R/\mathfrak{m}_R$.

Note:

(a) Γ_R is divisible.

(b) $\Gamma_R = \{0\}$ if and only if R is archimedean.

(c) κ_R is an archimedean real closed field and can therefore be uniquely embedded into the reals.

4.2. Power series fields over the reals:

Let Γ be an ordered abelian group that is divisible.

Let $\mathcal{R} := \mathbb{R}((t^\Gamma))$ be the **power series field** over $\mathbb{R}$ with respect to Γ. The elements of $\mathcal{R}$ are the formal power series $f = \sum_{\gamma\in\Gamma} a_\gamma t^\gamma$ with well-ordered support.

$\mathcal{R}$ is a real closed field (where t is infinitesimally positive) and can be made in a natural way into a model of the theory T_{an} of $\mathbb{R}_{\mathrm{an}}$.

Example: The **partial logarithm** on $\mathcal{R}$ is defined as follows.

Let $f \in \mathbb{R}_{>0} + \mathfrak{m}_\mathcal{R}$. Then $f = c(1+h)$ with $c \in \mathbb{R}_{>0}$ and $h \in \mathfrak{m}_\mathcal{R}$.

Let $\log(f) = \log(c) + L(h)$ where

$$L = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

is the logarithmic series.

4.3 Field of logarithmic-exponential series

The field of logarithmic exponential series over $\mathbb{R}$ is denoted by $\mathbb{R}((t))^{\mathrm{LE}}$. It consists of formal expressions of the form

$$t^{-1}e^{1/t}+e^{1/t}+t^{-1/2}-\log t+6+t+t^2+t^3+\ldots+e^{-1/t^2}+e^{-e^{1/t}}.$$

The following holds:

(a) $\mathbb{R}((t))^{\mathrm{LE}}$ is a model of the theory $T_{\mathrm{an,exp}}$ of $\mathbb{R}_{\mathrm{an,exp}}$.

(b) $\mathbb{R}((t))^{\mathrm{LE}}$ contains the T_{an}-model $\mathbb{R}((t^{\mathbb{R}}))$.

In particular the logarithm on $\mathbb{R}((t))^{\mathrm{LE}}$ extends the partial logarithm on $\mathbb{R}((t^{\mathbb{R}}))$. We have $\log(t^{-r}) = r\log(t^{-1})$ for all $r \in \mathbb{R}$.

5. Construction in the case of an archimedean value group

Recall: R is a real closed field, $a \in R^n$ and $\widehat{F}$ a function of the form

$$P\big(\varphi_1(x), \ldots, \varphi_r(x), \log(\varphi_1(x)), \ldots, \log(\varphi_r(x))\big)$$

where P is a polynomial and $\varphi_1, \ldots, \varphi_r$ are globally subanalytic on $\mathbb{R}^n$. We want to define $\widehat{F}(a)$ in a reasonable way.

Assume that the value group $\Gamma := \Gamma_R$ of R is archimedean.

Step 1: Embed R valuation preserving into $\mathbb{R}((t^\Gamma))$ (with respect to a section $\Gamma \to R^*$).

Step 2: Embed Γ order preserving into $\mathbb{R}$ (Hölder; uniquely determined by choosing an element of $\Gamma_{>0}$).

We obtain

$$R \hookrightarrow \mathbb{R}((t^\Gamma)) \hookrightarrow \mathbb{R}((t^\mathbb{R})) \hookrightarrow \mathbb{R}((t))^{\mathrm{LE}}.$$

We have that $\widehat{F}$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$. Since $\mathbb{R}((t))^{\mathrm{LE}}$ is a model of $T_{\mathrm{an,exp}}$, $\widehat{F}(a)$ is well-defined. By construction, $\widehat{F}$ is contained in $R_{\mathbf{LebAlg}} := \mathbb{R}((t^{\Gamma}))[X]$ where $X := \log(t^{-1})$ with $\mathbb{R} < X < t^{\Gamma_{<0}}$. We call this the **Lebesgue algebra** of R.

In the case that Γ is non-archimedean, the construction is more advanced. The theory of **Hahn groups** and LE-series fields over arbitrary models of $T_{\mathrm{an,exp}}$ are used.

We obtain for $n \in \mathbb{N}$ a **semialgebraic Lebesgue measure**

$$\lambda_{R,n} : \{\text{semialg. subsets of } R^n\} \to R_{\mathbf{LebAlg}} \cup \{+\infty\},$$

$$A \mapsto \lambda_{R,n}(A).$$

The construction is set up in such a way that it is uniquely determined by data that can be formulated completely in terms of the given real closed field. Moreover, in the case of an archimedean value group all these constructions are equivalent.

By a similar construction we obtain a **semialgebraic Lebesgue integral**

$$\mathrm{Int}_{R,n} : \{\text{integrable semial. subsets of } R^n\} \to R_{\mathbf{LebAlg}},$$

$$f \mapsto \int_{R^n} f(x)\, d\lambda_{R,n}(x).$$

Semialgebraic Lebesgue measure and the corresponding semialgebraic Lebesgue integral are intertwined in the familiar way:

$$\lambda_{R,n}(A) = \int \mathbb{1}_A(x)\, d\lambda_{R,n}(x).$$

Example:

Let $\mathbb{P}$ be the field of convergent or formal **Puiseux series** over $\mathbb{R}$. Then in any dimension measure and integration take their values in the **polynomial ring** $\mathbb{P}[X]$ (and $+\infty$).

6. Proof of the classical results

Basic idea:

- Formulate a property of classical Lebesgue measure or integration theory as a statement in the natural language of $\mathbb{R}_{\text{an,exp}}$.
- Transfer the result to the non-standard model of $T_{\text{an,exp}}$ (LE-series field).

This does the job for example for the additivity of the measure or for the transformation formula. For limit processes we are more ambitious to obtain results that can be applied.

Transformation formula:

Let U, V be open in R^n and let $\varphi : U \to V$ be a semialgebraic C^1-diffeomorphism. Let $f : V \to R$ be semialgebraic. Then f is integrable over V if and only if $(f \circ \varphi)|\det(D_\varphi)|$ is integrable over U, and in this case

$$\int_V f\, d\lambda_n = \int_U (f \circ \varphi)|\det(D_\varphi)|\, d\lambda_n.$$

Lebesgue's theorem on dominated convergence:
Let $f : R^{n+1} \to R, (t,x) \mapsto f(t,x) = f_t(x)$, be semialgebraic. Assume that there is some integrable semialgebraic function $h : R^n \to R$ such that $|f_t| \leq |h|$ for all $t \in R_{>0}$. Then $\lim_{t\to+\infty, t\in R} f_t$ is integrable and

$$\int \lim_{t\to+\infty, t\in R} f_t \, d\lambda_n = \lim_{t\to+\infty, t\in R} \int f_t \, d\lambda_n.$$

Here $\lim_{t\to+\infty, t\in R} f_t$ is the limit of semialgebraic functions $R^n \to R$ and $\lim_{t\to+\infty, t\in R} \int f_t \, d\lambda_n$ is the limit in $R_{\mathbf{LebAlg}}$ as $t \to 0$ in R and **not** a priori in the big $T_{\mathrm{an,exp}}$-model.

This needs o-minimal analysis.

7. Integration over Nash manifolds

Nash manifolds:

1) One defines Nash manifolds (and more general Nash manifolds with boundary) over an arbitrary real closed field R by an atlas consisting of finitely many charts such that the transition maps are Nash diffeomorphisms. These are semialgebraic spaces in the spirit of Delfs & Knebusch.

2) Then one defines integration of Nash differential forms over semialgebraic subsets of oriented Nash manifolds with boundary.

Example:

Let $\varphi = (x_1, \ldots, x_k) : U \to R^n$ be an orientation preserving chart of the n-dimensional oriented Nash manifold M with boundary. Then a Nash differential form ω of degree n on U is given by

$$\omega(x) = f(x)\, dx_1 \wedge \ldots \wedge dx_n$$

where $f : U \to R$ is Nash. For a semialgebraic subset A of U set

$$\int_A \omega := \int_{\varphi(A)} f\left(\varphi^{-1}(t)\right) d\lambda_{R,n}(t)$$

if $\int_{\varphi(A)} \left|f\left(\varphi^{-1}(t)\right)\right| d\lambda_{R,n}(t) < \infty$.

Definition:

Let M be a Nash manifold with boundary and let $A \subset M$ semialgebraic. Then A is called **complete**, if for all semialgebraic curves $\gamma :]0,1[\to A$ we have that $\lim_{t \to 0} \gamma(t) \in A$.

Remark:

In the case $M = R^n$ we have that A is complete if and only if A is bounded and closed.

Theorem:

Let M be an n-dimensional Nash manifold with boundary and let A be a complete semialgebraic subset of M of the same dimension. Then there is a largest n-dimensional Nash submanifold N_A of M that is contained in A.

Definition:

Let

$$\mathrm{bd}(A) := \text{boundary of } N_A \text{ (in the sense of manifolds).}$$

Remark:

We have $\mathrm{bd}(A) = \partial A$ if the topological boundary ∂A of A is smooth.

Stokes' theorem:

Let M be an oriented n-dimensional Nash manifold with boundary and let A be a complete semialgebraic subset of M of the same dimension. Let ω be a Nash differential form on M of degree $n-1$. Then

$$\int_A d\omega = \int_{\mathrm{bd}(A)} \omega.$$

- Semialgebraic version of Stokes' theorem on the reals
- Extension to arbitrary dimensions of A

Riemannian Nash manifolds:

1) One defines Riemannian Nash manifolds as in the classical case.

2) One defines the canonical volume form ω_M for an oriented Riemannian Nash manifold M.

3) This gives a surface measure on the oriented Riemannian Nash manifold.

Example:

Let M be an oriented n-dimensional Nash manifold and let $\varphi = (x_1, \ldots, x_n) : U \to R^n$ be an orientation preserving chart. Then on U

$$\omega_M = \sqrt{g}\, dx_1 \wedge \ldots \wedge dx_n$$

where $g = \det(\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle)_{i,j}$. For a semialgebraic subset A of M contained in U we have

$$\mathrm{vol}_k(A) = \int_A \omega_M = \int_{\varphi(A)} (\sqrt{g} \circ \varphi^{-1})\, d\lambda_{R,k}.$$

Consequence:

Let $\mathbb{P}$ be the field of Puiseux series over $\mathbb{R}$. Let $A \subset \mathbb{P}^n$ be semialgebraic of dimension k. One obtains a k-dimensional surface measure

$$\mathrm{vol}_k : \{\text{Semialgebr. subsets of } A\} \to \mathbb{P}[X] \cup \{\infty\},$$

$$B \mapsto \mathrm{vol}_k(B).$$

Theorem:

Let $A \subset \mathbb{P}^n$ be semialgebraic of dimension k. If A is bounded then $\mathrm{vol}_k(A) < \infty$.