The Asymptotic Behaviour of the Riemann Mapping Function at Analytic Cusps

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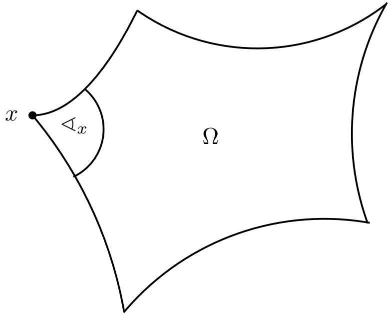
Theorem (T. Kaiser 2009)

Let $\Omega \subsetneq \mathbb{C}$ be bounded, simply connected, and semianalytic. Assume that the opening angle $\sphericalangle_x$ is an irrational multiple of π for all singular boundary points $x \in \partial\Omega$. Then $\phi : \Omega \to \mathbb{E}$ is definable in an o-minimal structure.



General Premises

- Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain with piecewise analytic boundary.
- Let $0 \in \partial\Omega$ be a singular boundary point.
- Let $\varphi : \Omega \to \mathbb{H}$ be a Riemann map with $\varphi(0) = 0$.

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Motivation

General Premises

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Theorem (R. S. Lehman, 1957)

Assume that the opening angle at $0 \in \partial\Omega$ is $\pi\alpha$ with $0 < \alpha \leq 2$.

(a) *If $\alpha \notin \mathbb{Q}$ then φ has an asymptotic power series expansion at 0 of the following kind*

$$\sum_{k \geq 0,\ l \geq 1} a_{k,l} z^{k + \frac{l}{\alpha}},$$

where $a_{k,l} \in \mathbb{C}$ and $a_{0,1} \neq 0$, i.e.

$$\varphi(z) - \sum_{k + \frac{l}{\alpha} \leq N} a_{k,l} z^{k + \frac{l}{\alpha}} = o(z^N)$$

for all $N \in \mathbb{N}$.

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(b) *If* $\alpha = \frac{p}{q}$, *with p and q coprime, then* φ *has an asymptotic power series expansion at* 0 *of the following kind*

$$\sum_{k\geq 0,\ 1\leq l\leq q,\ 0\leq m\leq \frac{k}{p}} a_{k,l,m} z^{k+\frac{l}{\alpha}} (\log(z))^m$$

where $a_{k,l,m} \in \mathbb{C}$ *and* $a_{0,1,0} \neq 0$.

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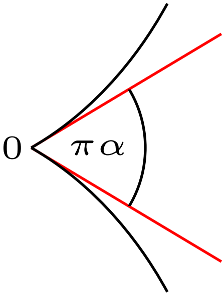
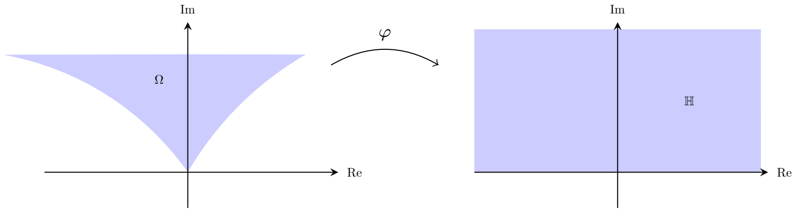
Definition (Analytic Corner)

We say that Ω has an analytic corner at 0 if 0 is a singular boundary point and the boundary at 0 is locally given by two regular analytic arcs with opening angle $\pi\alpha$ where $0 < \alpha \leq 2$.



Analytic Corners



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Theorem (L. Lichtenstein (1911), S. Warschawski (1955))

At an analytic corner at 0 with opening angle $\pi\alpha$ with $0 < \alpha \leq 2$ we have at 0 on Ω

(a) $\varphi(z) \sim z^{\frac{1}{\alpha}}$

(b) $\varphi'(z) \sim z^{\frac{1}{\alpha}-1}$

(c) $\varphi^{(n)}(z) \begin{cases} \sim z^{\frac{1}{\alpha}-n} & \text{for } \alpha \neq \frac{1}{k}, k \in \mathbb{N} \\ = O(z^{\frac{1}{\alpha}-n}) & \text{for } \alpha = \frac{1}{k}, k \in \mathbb{N} \end{cases}$ *for* $n \geq 2$

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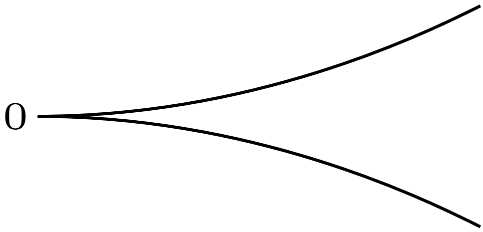
Definition (Analytic Cusp)

We say that Ω has an analytic cusp at 0 if 0 is a singular boundary point and the boundary at 0 is locally given by two regular analytic arcs such that the opening angle vanishes.



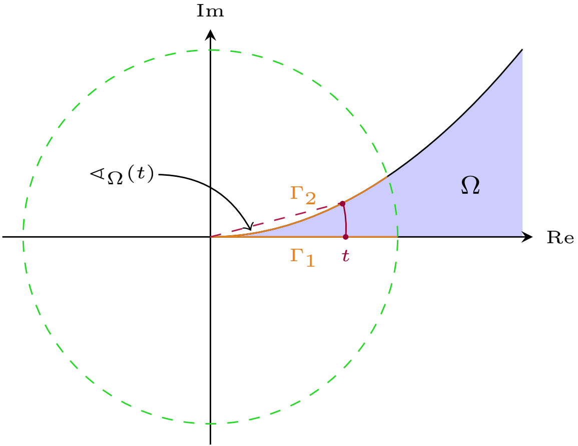
Setting

After applying a coordinate transformation we can assume that locally the boundary of Ω is given by the arcs Γ_1 and Γ_2 with the parameterisations

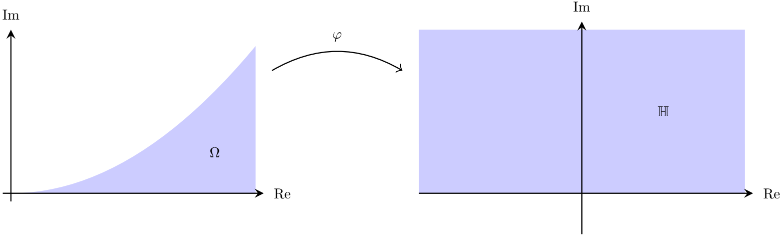
$$\gamma_1(t) = t \text{ and } \gamma_2(t) = t\exp(i\sphericalangle_\Omega(t)),$$

resp. Hereby, $\sphericalangle_\Omega(t) = \sum_{k=d}^{\infty} a_k t^k$ is a real power series with $d \in \mathbb{N}$ and $a_d \neq 0$.

Analytic Cusps



Analytic Cusps



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Theorem

We have at 0 *on* Ω

$$\varphi(z) \sim \exp\left(\sum_{n=0}^{d-1} b_n z^{n-d} + a\log(z)\right)$$

with

$$b_n := \frac{\pi c_n}{n-d} \text{ and } a := \pi c_d,$$

where c_k *are the coefficients of the Laurent series*

$$\frac{1}{\sphericalangle_\Omega(t)} = t^{-d}\sum_{k=0}^{\infty} c_k t^k.$$

Example

Let

$$\Omega := \left\{ z \in \mathbb{C} \mid 0 < |z| < \frac{1}{2}, 0 < \arg(z) < |z| - |z|^2 \right\}$$

then

$$\varphi(z) \sim \exp\left(-\frac{\pi}{z} + \pi \log(z) \right).$$

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Remark

If $a_{d+1} = \ldots = a_{2d} = 0$ we have

$$\varphi(z) \sim \exp\left(-\frac{\pi}{a_d d z^d}\right)$$

at 0 on Ω.

Recall: $\sphericalangle_\Omega(t) = \sum_{k=d}^{\infty} a_k t^k$

Analytic Cusp

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Example

Let

$$\Omega := \left\{ z \in \mathbb{C} \mid 0 < |z| < \frac{1}{2}, 0 < \arg(z) < a_d |z|^d \right\}$$

then

$$\varphi(z) \sim \exp\left(-\frac{\pi}{a_d d z^d} \right).$$

Theorem

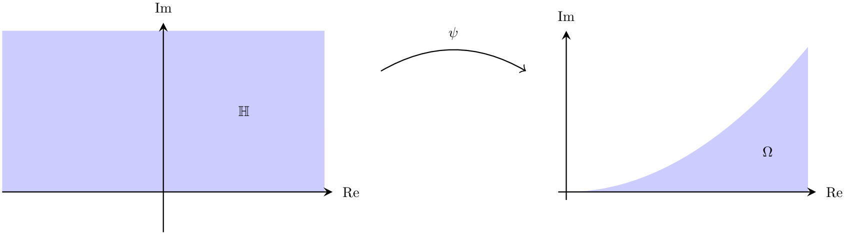
We have for $n \in \mathbb{N}$

$$\varphi^{(n)}(z) \sim \exp\left(\sum_{k=0}^{d-1} b_k z^{k-d} + a\log(z)\right) z^{-n(d+1)}$$

at 0 *on* Ω.

Analytic Cusps

Inverse function ψ



Theorem

Let $\psi : \mathbb{H} \to \Omega$ *be a conformal map with* $\psi(0) = 0$*. Then*

$$\psi(z) \simeq \left(-\frac{\pi}{a_d d \log(z)}\right)^{\frac{1}{d}}$$

at 0 *on* $\mathbb{H}$*.*

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Theorem

We have for $n \in \mathbb{N}$

$$\psi^{(n)}(z) \sim \left(-\frac{1}{\log(z)}\right)^{\frac{1}{d}+1} z^{-n}$$

at 0 on $\mathbb{H}$.

Content

Conclusion

Contributions

Asymptotic behaviour at analytic cusps of

- $\varphi : \Omega \to \mathbb{H}$
- $\varphi^{(n)}$
- $\psi : \mathbb{H} \to \Omega$
- $\psi^{(n)}$

Open Questions

- Development of the mapping function in a generalized power series?
- O-minimality?

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Open Questions

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Thank you!