Lebesgue integration of oscillating and subanalytic functions

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(joint work with R. Cluckers, G. Comte, D. Miller, J.-P. Rolin)

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Motivation and background

Oscillatory integrals. $\lambda \in \mathbb{R},\ x = (x_1, \ldots, x_n) \in \mathbb{R}^n$

$$\mathcal{I}(\lambda) = \int_{\mathbb{R}^n} e^{\mathrm{i}\lambda\varphi(x)}\psi(x)\,\mathrm{d}x, \text{ where:}$$

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$n > 1$ reduce to the case $n = 1$ by *monomializing* the phase (res. of sing.). Suitable blow-ups act as changes of variables in $\mathbb{R}^n$, outside a set of measure 0. Using Fubini and the case $n = 1$, one proves:

$$\mathcal{I}(\lambda) \sim e^{\mathrm{i}\lambda\varphi(0)} \sum_{q\in\mathbb{Q}} \sum_{k=0}^{n-1} a_{q,k}(\psi)\,\lambda^q (\log\lambda)^k .$$

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Proviso. For the rest of the talk, *subanalytic* means "globally subanalytic".

Our framework: parametric integrals and subanalytic functions

Def. For $X \subseteq \mathbb{R}^m$ and $f : X \times \mathbb{R}^n \to \mathbb{R}$, define, $\forall x \in X$ s.t. $f(x, \cdot) \in L^1(\mathbb{R}^n)$,

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The answer is **NO**:
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i.e. Si $\sim$ to a polynomial in $\{\cos x,\sin x\}$ with coefficients *divergent series* $\in\mathbb{R}\left[\!\left[\frac{1}{x}\right]\!\right]$.

Oscillating and subanalytic functions

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the $\mathbb{C}$- algebra $\mathcal{D}(X)$ generated by $\left\{ g(x),\ \log h(x),\ e^{i\varphi(x)} :\ g, h, \varphi \in \mathcal{S}(X) \right\}$ is **not** stable under parametric integration.

Example. $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t} \mathrm{d}t = \int_{\mathbb{R}} \frac{\chi_{[0,x]}(t)}{2it} \left(e^{it} - e^{-it} \right) \mathrm{d}t \notin \mathcal{D}(\mathbb{R}^+)$. Why?

It is well-known that

$$\mathrm{Si}(x) \underset{x \to +\infty}{\sim} \frac{\pi}{2} - \frac{\cos x}{x} \sum_{k \geq 0} (-1)^k \frac{(2k)!}{x^{2k}} - \frac{\sin x}{x} \sum_{k \geq 0} (-1)^k \frac{(2k+1)!}{x^{2k+1}},$$

i.e. Si $\sim$ to a polynomial in $\{\cos x, \sin x\}$ with coefficients *divergent series* $\in \mathbb{R}\left[\!\left[\frac{1}{x} \right]\!\right]$.

However, if $f \in \mathcal{D}(\mathbb{R}^+)$, then f is asymptotic to a polynomial in $\{\log x\} \cup \{\cos(c_j x^{r_j}), \sin(c_j x^{r_j})\}_{j=1}^N$ with *convergent coefficients* $\in \mathbb{R}\left\{ x^{-\frac{1}{d}} \right\}$,
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To see this:

- For $g\in\mathcal{S}\left(\mathbb{R}^+\right)$, $\exists c\in\mathbb{R},r\in\mathbb{Q},d\in\mathbb{N}$, $\exists H\in\mathbb{R}\{Y\}^*$ s.t. $g(x)=cx^rH\left(x^{-\frac{1}{d}}\right)$.

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- $\log(g(x)) = r \log x + \log\left(cH\left(x^{-\frac{1}{d}}\right)\right)$, $e^{ig(x)} = e^{i \sum_{j \leq rd} c_j x^{r - j/d}} \cdot e^{i \sum_{j > rd} c_j x^{r - j/d}}$.

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- if g is *bounded*, then $\log(g(x))$, $\cos(g(x))$, $\sin(g(x))\in\mathcal{S}\left(\mathbb{R}^+\right)$.

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One-dimensional transcendentals

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Corollary. $\mathcal{E} = \bigcup \mathcal{E}(X)$ is the smallest collection of $\mathbb{C}$-algebras containing $\mathcal{S} \cup \{e^{\mathrm{i}\varphi} :\ \varphi \in \mathcal{S}\}$ and stable under parametric integration.
Moreover, $\mathcal{E}$ is closed under taking Fourier transforms.

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Def. A generator $T(x,y) \in \mathcal{E}(X \times \mathbb{R}^n)$ is ***naive in* y** if γ does not depend on y.

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Preparation Theorem. Given $f \in \mathcal{E}(X \times \mathbb{R})$, up to cell decomposition of $X \times \mathbb{R}$, there are finite index sets $J^{\text{Int}}, J^{\text{Naive}} \subseteq \mathbb{N}$ and generators T_j, S_j s.t.

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The case $n > 1$ follows by Fubini and induction on n. □

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Lemma. If $F : \mathbb{R}^n \to \mathbb{R}$ is almost periodic and $G(y) = F\left(y, y^2, \ldots, y^n\right)$, then $\exists \varepsilon > 0$ s.t. the set $V_\varepsilon := \{y : \; |G(y)| \geq \varepsilon\}$ intersects every interval of size ν (for some $\nu > 0$), and such an intersection has measure $\geq \delta$ (for some $\delta > 0$).

Example. $f(x) = \sin(2\pi x) + \sin\left(2\sqrt{2}\pi x\right)$ is not periodic. However, $\forall \varepsilon > 0 \ \exists \ \infty$ many τ s.t. $x \in \mathbb{R} \ |f(x+\tau) - f(x)| < \varepsilon$.

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Recall: we have $G(y) = \sum_{j \in J} f_j e^{\mathrm{i} p_j(y)}$, which is not almost periodic, and we want to prove that $\displaystyle\int_{V_\varepsilon} \frac{1}{y} \mathrm{d}y = \infty$.

Almost periodic functions

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Apply the above lemma to $F(x) = \sum_{j \in J} f_j e^{\mathrm{i} L_j(x)}$, where $L_j(x_1, \ldots, x_n)$ is the linear form such that $p_j(y) = L_j\left(y, y^2, \ldots, y^n\right)$. $\square$